Positivity, sums of squares and the multi-dimensional moment problem II *

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Abstract

The paper is a continuation of work initiated by the first two authors in [K–M]. Section 1 is introductory. In Section 2 we prove a basic lemma, Lemma 2.1, and use it to give new proofs of key technical results of Scheiderer in [S1] [S2] in the compact case; see Corollaries 2.3, 2.4 and 2.5. Lemma 2.1 is also used in Section 3 where we continue the examination of the case $n = 1$ initiated in [K–M], concentrating on the compact case. In Section 4 we prove certain uniform degree bounds for representations in the case $n = 1$, which we then use in Section 5 to prove that (‡) holds for basic closed semi-algebraic subsets of cylinders with compact cross-section, provided the generators satisfy certain conditions; see Theorem 5.3 and Corollary 5.5. Theorem 5.3 provides a partial answer to a question raised by Schmüdgen in [Sc2]. We also show that, for basic closed semi-algebraic subsets of cylinders with compact cross-section, the sufficient conditions for (SMP) given in [Sc2] are also necessary; see Corollary 5.2(b). In Section 6 we prove a module variant of the result in [Sc2], in the same spirit as Putinar's variant [Pu] of the result in [Sc1] in the compact case; see Theorem 6.1. We apply this to basic closed semi-algebraic subsets of cylinders with compact cross-section; see Corollary 6.4. In Section 7 we apply the results from Section 5 to solve two of the open problems listed in [K–M]; see Corollary 7.1 and Corollary 7.4. In Section 8 we consider a number of examples in the plane. In Section 9 we list some open problems.

1 Introduction

Let $\mathbb{R}[X]$ denote the polynomial ring in n variables $X = (X_1, \ldots, X_n)$, with real coefficients, and consider a finite set $S = \{g_1, \cdots, g_s\}$ of polynomials in $\mathbb{R}[X]$. We

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denote by K_S the basic closed semi-algebraic set in $\mathbb{R}^n$ defined by:

$$K_S = \{x \in \mathbb{R}^n \ : \ g_1(x) \geq 0, \ldots, g_s(x) \geq 0\}.$$

We denote by T_S the preordering in $\mathbb{R}[X]$ generated by S:

$$T_S = \{ \sum_{e \in \{0,1\}^s} \sigma_e g^e \ : \ \sigma_e \in \sum \mathbb{R}[X]^2\},$$

where $g^e := g_1^{e_1} \cdots g_s^{e_s}$, if $e = (e_1, \cdots, e_s)$, and $\sum \mathbb{R}[X]^2$ denotes the preordering of $\mathbb{R}[X]$ consisting of sums of squares. Note: $K_\emptyset = \mathbb{R}^n$, $T_\emptyset = \sum \mathbb{R}[X]^2$. By the Positivstellensatz [St], $K_S = \emptyset \Leftrightarrow -1 \in T_S \Leftrightarrow T_S = \mathbb{R}[X]$. Set

$$T_S^{\text{alg}} = \{f \in \mathbb{R}[X] : f \geq 0 \text{ on } K_S\},$$

$$T_S^{\vee} = \{L : \mathbb{R}[X] \to \mathbb{R} : L \text{ is linear } (\neq 0) \text{ and } L(T_S) \geq 0\},$$

and

$$T_S^{\text{lin}} = T_S^{\vee\vee} = \{f \in \mathbb{R}[X] : L(f) \geq 0 \text{ for all } L \in T_S^{\vee}\}.$$

Note: We have the inclusions $T_S \subseteq T_S^{\text{lin}} \subseteq T_S^{\text{alg}}$. The sets $T_S^{\vee}$, T_S^{lin} depend on T_S. The set T_S^{alg} depends only on K_S.

The set T_S^{alg} is a preordering, called the **saturation** of T_S. We say T_S is **saturated** if $T_S^{\text{alg}} = T_S$. The set T_S^{lin} is the **closure** of T_S in $\mathbb{R}[X]$, giving $\mathbb{R}[X]$ the unique finest locally convex topology [Po–S, p. 76]. The set T_S^{lin} is a preordering [Po–S, Lem. 1.2]. We say T_S is **closed** if $T_S^{\text{lin}} = T_S$. This holds in a variety of special cases, e.g., if K_S contains an n-dimensional cone [K–M, Th. 3.5] [Po–S, Prop. 3.7]. It also fails in many cases, e.g., if K_S is compact and $\dim(K_S) \geq 3$. This follows from [S1, Prop 6.1] using [Sc1, Cor. 2].

The general Moment Problem is the following: For a linear functional L on $\mathbb{R}[X]$, when is there a positive Borel measure μ or $\mathbb{R}^n$ such that $\forall f \in \mathbb{R}[X]$ $L(f) = \int_{\mathbb{R}^n} f d\mu$? The following result is due to Haviland [H1] [H2]. For a proof based on the Riesz representation theorem, see [M2, Th. 3.1], for example.

Theorem 1.1 *For a linear functional L on $\mathbb{R}[X]$ and a closed set K of $\mathbb{R}^n$, the following are equivalent:*

(i) $\exists$ a positive Borel measure μ on K such that $\forall f \in \mathbb{R}[X], L(f) = \int_K f d\mu$.

(ii) $\forall$ $f \in \mathbb{R}[X]$, $f \geq 0$ on $K \Rightarrow L(f) \geq 0$.

Since T_S^{alg} is not finitely generated in general , one is interested in approximating it by T_S. Therefore one studies the following concrete Moment Problem: When is it true that every $L \in T_S^{\vee}$ comes from a positive Borel measure on K_S? By Theorem 1.1, this is equivalent to asking when the following condition holds:

(SMP) $$T_S^{\text{alg}} = T_S^{\text{lin}}.$$

In the landmark paper [Sc1], Schmüdgen proves (SMP) when the basic closed semi-algebraic set K_S is compact. Moreover, for compact K_S, he gets the following substantial improvement of the Positivstellensatz:

$$(\dagger) \qquad \forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K_S \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in T_S.$$

See [B–W], [M1] or [P–D] for Wörmann's proof of (†), which is based on the Kadison-Dubois representation theorem.

We turn now to the case where K_S is non-compact. Given a pair of sets S, S' with $K_S \subseteq K_{S'}$, one can also ask: When is it true that every $L \in T_S^\vee$ comes from a positive Borel measure supported by $K_{S'}$? By Theorem 1.1 this is equivalent to asking if $T_{S'}^{\text{alg}} \subseteq T_S^{\text{lin}}$. We will consider this property in the special case $S' = \emptyset$:

$$(\text{MP}) \qquad T_\emptyset^{\text{alg}} \subseteq T_S^{\text{lin}}.$$

Note: T_S is saturated ⇒ (†) holds ⇒ (SMP) holds ⇒ (MP) holds.

Proposition 1.2 *If K_S contains a cone of dimension 2 then (MP) fails.*

Note: The statement of Proposition 1.2 is stronger than [K–M, Cor. 3.10], but the proof is the same. Namely, one uses [K–M, Th. 3.5] along with [S1, Remark 6.7] to produce $p \in T_\emptyset^{\text{alg}}$, $p \notin T_S^{\text{lin}}$. See [Po–S, Cor. 3.10] for a more general criterion.

Note: For $n \geq 3$ the condition that K_S contains a cone of dimension 2 does not imply that T_S is closed.

Example 1.3 Let $n \geq 1$ and pick S_1 in $\mathbb{R}[X]$ such that K_{S_1} is compact and T_{S_1} is not closed. Define S in $\mathbb{R}[X, Y, Z]$ by $S = S_1$. Then $K_S = K_{S_1} \times \mathbb{R}^2$ contains a 2-dimensional cone. Also $T_{S_1}^{\text{lin}} \subseteq T_S^{\text{lin}}$ and $T_{S_1} = T_S \cap \mathbb{R}[X]$ so if $p \in T_{S_1}^{\text{lin}}$, $p \notin T_{S_1}$, then $p \in T_S^{\text{lin}}$, $p \notin T_S$. The possible choices for S_1 are necessarily a bit artificial if $n = 1$, e.g., $S_1 = \{X^3(1 - X)\}$ will do. For $n = 2$ it seems that less contrived choices should exist. For $n \geq 3$ any S_1 with $\dim(K_{S_1}) \geq 3$ will do [S1, Prop. 6.1].

In [K–M], the authors consider the intermediate condition:

$$(\ddagger) \quad \forall\ f \in \mathbb{R}[x], f \geq 0 \text{ on } K_S \Rightarrow \exists q \in \mathbb{R}[X] \text{ such that } \forall \text{ real } \epsilon > 0,\ f + \epsilon q \in T_S.$$

It is shown that (‡) is strictly weaker than (†), but implies (SMP). The condition (‡) holds in various non-compact cases, e.g., for basic closed semi-algebraic sets obtained by a certain natural dimension extension process [K–M, Cor. 4.5] and for cylinders with compact cross-section [K–M, Th. 5.1]. Define

$$T_S^\ddagger = \{f \in \mathbb{R}[X] \ :\ \exists q \in \mathbb{R}[X] \text{ such that } \forall \text{ real } \epsilon > 0\,,\ f + \epsilon q \in T_S\}.$$

Note: (i) $T_S \subseteq T_S^\ddagger \subseteq T_S^{\text{lin}}$. (ii) (‡) holds iff $T_S^{\text{alg}} = T_S^\ddagger$. (iii) It follows from the proof of [K–M, Cor. 4.2] that the element q appearing in the description of

$T_S^\ddagger$ can be chosen from the set $\{p^\ell : \ell \geq 0\}$, where $p \in 1 + T_S$ is any polynomial which 'grows sufficiently rapidly' on K_S, e.g., $p = 1 + \|X\|^2$ always works. Here, $\|X\|^2 := \sum_{i=1}^n X_i^2$. If K_S is compact then $p = 1$ works.

Denote by P_d the (finite dimensional) vector space consisting of all polynomials in $\mathbb{R}[X]$ of degree $\leq 2d$, and by $T_d = T_S \cap P_d$. The set T_d is obviously a cone in P_d, i.e., $T_d + T_d \subseteq T_d$ and $\mathbb{R}^+ T_d \subseteq T_d$. Denote by $\overline{T}_d$ the closure of T_d in P_d.

Proposition 1.4 *(i)* $T_S^\ddagger = \cup_{d \geq 0} \overline{T}_d$. *(ii)* $T_S^\ddagger$ *is a preordering.*

Proof: Assertion (i) is clear from the proof of [K–M, Prop. 1.3]. For polynomials f, g, the coefficients of $f + g$ and fg are polynomial functions (in particular, continuous functions) of the coefficients of f and of g. Assertion (ii) is clear from this, using (i). □

Observe that (†), (‡), (SMP) and (MP) all depend on T_S. Consequently, if (†), resp., (‡), resp., (SMP), resp., (MP) holds then we say that (†), resp., (‡), resp., (SMP), resp., (MP) **holds** for T_S or that T_S **satisfies** (†), resp., (‡), resp., (SMP), resp., (MP).

In Section 2 we prove Lemma 2.1, which we need later, in Section 3, but which is also of independent interest in that it yields alternate proofs of key technical results in [S1] [S2]. In Sections 3 and 4 we continue the analysis of the 1-dimensional case initiated in [K–M]. The results in these two sections are of interest in their own right, but they are also essential in dealing with the fiber sets that arise in the study of **subsets** of cylinders with compact cross-section.

In [Sc2], Schmüdgen considers the following general set-up: Suppose $h_1, \ldots, h_d \in \mathbb{R}[X]$ are bounded on K_S. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ define

$$S_\lambda = S \cup \{h_1 - \lambda_1, -(h_1 - \lambda_1), \ldots, h_d - \lambda_d, -(h_d - \lambda_d)\}.$$

Thus $K_{S_\lambda} = K_S \cap C_\lambda$ where C_λ denotes the algebraic set in $\mathbb{R}^n$ defined by the equations $h_i(X) = \lambda_i$, $i = 1, \ldots, d$. By the assumption $K_{S_\lambda} = \emptyset$ (i.e., $T_{S_\lambda} = \mathbb{R}[X]$) for $\|\lambda\|$ sufficiently large. Schmüdgen proves:

Theorem 1.5 *Suppose* $h_1, \ldots, h_d \in \mathbb{R}[X]$ *are bounded on* K_S. *If (SMP) (resp., (MP)) holds for* T_{S_λ} *for each* λ *then (SMP) (resp. (MP)) holds for* T_S.

Theorem 1.5 applies, in particular, to subsets of cylinders with compact cross-section. In this special case, we prove that the converse of Theorem 1.5 also holds; see Corollary 5.2.

The question of whether (SMP) is strictly weaker than (‡) is posed as an open problem in [K–M]. It seems likely that this is the case, but no examples are known. In fact, no examples are known where $T_S^\ddagger \neq T_S^{\text{lin}}$. In [Sc2], Schmüdgen asks the following related question: Is it true that Theorem 1.5 continues to hold with (SMP) replaced by (‡)?

We provide a partial answer to Schmüdgen's question in the case of subsets of cylinders with compact cross-section; see Theorem 5.3. At the same time, we produce a variety of concrete examples where (SMP) holds but where we are unable to prove (‡). Our results are applied to settle two open problems in [K–M]; see Corollary 7.1 and Theorem 7.4.

As explained in [K–M], one can also consider the (generally smaller) quadratic module

$$M_S = \{\sum_{i=0}^{s} \sigma_i g_i \; : \; \sigma_i \in \sum \mathbb{R}[X]^2\},$$

generated by S (here $g_0 := 1$) and the corresponding objects

$$M_S^{\vee} = \{L : \mathbb{R}[X] \to \mathbb{R} \mid L \text{ is linear } (\neq 0) \text{ and } L(M_S) \geq 0\},$$

$$M_S^{\text{lin}} = M_S^{\vee\vee} = \{f \in \mathbb{R}[X] : L(f) \geq 0 \text{ for all } L \in M_S^{\vee}\} \text{ and}$$

$$M_S^{\ddagger} = \cup_{d \geq 0} \overline{M}_d \quad \text{where} \quad M_d := M_S \cap P_d.$$

M_S^{lin} is the closure of M_S in $\mathbb{R}[X]$. One can also consider the corresponding conditions

(SMP′) $$T_S^{\text{alg}} = M_S^{\text{lin}},$$

(MP′) $$T_\emptyset^{\text{alg}} \subseteq M_S^{\text{lin}},$$

(†′) $$\forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K_S \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in M_S \quad \text{and}$$

(‡′) $\forall\ f \in \mathbb{R}[X], f \geq 0$ on $K_S \Rightarrow \exists q \in \mathbb{R}[X]$ such that $\forall$ real $\epsilon > 0,\ f + \epsilon q \in M_S$.

Again, (MP′) is strictly weaker than (SMP′) and (‡′) is strictly weaker than (†′), but implies (SMP′). If (†′), resp., (‡′), resp., (SMP′), resp., (MP′), holds then we say that (†), resp., (‡), resp., (SMP), resp., (MP) **holds** for M_S or that M_S **satisfies** (†), resp., (‡), resp., (SMP), resp., (MP).

According to Putinar [Pu], also see [J], if K_S is compact and there exists N such that $N - \|X\|^2 \in M_S$, then (†) holds for M_S. In certain cases the condition that there exists N such that $N - \|X\|^2 \in M_S$ is automatically fulfilled; see [J–P]. In [M2] various results in [K–M], in particular, the results for dimension extension and for cylinders with compact cross-section, are extended under suitable assumptions to quadratic modules; see [M2, Cor. 4.3, Cor 5.3].

Insofar as it is possible, we develop quadratic module versions of the various results considered. In particular, we prove quadratic module versions of Theorem 1.5 and of Theorem 5.3; see Theorem 6.1 and Corollary 5.5.

2 Basic Lemma

The following general result is useful:

Lemma 2.1 *Let C be a compact Hausdorff space, A a commutative ring with 1 with $\frac{1}{n} \in A$ for some integer $n \geq 2$ and $\phi : A \to Cont(C, \mathbb{R})$ a (unitary) ring homomorphism. Suppose f, $g \in A$ are such that $\phi(f) \geq 0$, $\phi(g) \geq 0$ and $sf + tg = 1$ for some $s, t \in A$. Then there exist $\sigma, \tau \in A$ such that $\sigma f + \tau g = 1$ and $\phi(\sigma), \phi(\tau)$ are strictly positive.*

Proof: We suppress ϕ from the notation. Let $s, t \in A$ be such that $1 = sf + tg$. On the compact set

$$L_1 := \{p \in C \mid s(p) \leq 0\},$$

$tg = 1 - sf \geq 1$. Thus $g > 0$ on L_1 so, for N sufficiently large, $s + Ng > 0$ on L_1. On $C \backslash L_1$ this is obviously also true. Define $s_1 = s + Ng$, $t_1 = t - Nf$. Thus $1 = s_1 f + t_1 g$ in A and $s_1 > 0$ on C. Choose a positive rational $\delta \in A$ so small that $\delta fg < 1$ on C. Choose a positive rational $\epsilon \in A$ so small that, on the compact set

$$L_2 = \{p \in C : g(p) \leq \epsilon\},$$

$f > 0$, $1 > \epsilon\delta f$ and $\epsilon t_1 + s_1 f > 0$. Choose k so large that, on the set L_2, $\epsilon t_1 + s_1 f > s_1 f(1 - \epsilon\delta f)^k$ and, on the set

$$L_3 = \{p \in C : g(p) \geq \epsilon\},$$

$s_1 f(1-\epsilon\delta f)^k < 1$. Choose $r = s_1 \delta f \sum_{i=0}^{k-1}(1-\delta fg)^i$. Choose $\sigma = s_1 - rg$, $\tau = t_1 + rf$. Thus $1 = \sigma f + \tau g$ in A. It remains to verify that $\sigma, \tau > 0$ on C. Using the identity $(1 - z)\sum_{i=0}^{k-1} z^i = 1 - z^k$, we see that, on C,

$$\begin{aligned} \sigma &= s_1 - rg = s_1 - s_1 \delta fg \sum_{i=0}^{k-1} (1 - \delta fg)^i \\ &= s_1 - s_1(1 - (1 - \delta fg)^k) = s_1(1 - \delta fg)^k > 0. \end{aligned}$$

On L_2,

$$\begin{aligned} \tau &= t_1 + rf = t_1 + s_1 \delta f^2 \sum_{i=0}^{k-1} (1 - \delta fg)^i \\ &\geq t_1 + s_1 \delta f^2 \sum_{i=0}^{k-1} (1 - \epsilon\delta f)^i = t_1 + (s_1 f/\epsilon)(1 - (1 - \epsilon\delta f)^k) > 0. \end{aligned}$$

On L_3,

$$\begin{aligned} \tau &= t_1 + rf = t_1 + (f/g)rg = t_1 + (f/g)s_1(1 - (1 - \delta fg)^k) \\ &= t_1 + s_1 f/g - (s_1 f/g)(1 - \delta fg)^k = 1/g - (s_1 f/g)(1 - \delta fg)^k > 0. \end{aligned}$$

This completes the proof. □

We apply Lemma 2.1 to $C = K_S$ and $A = \mathbb{R}[X]/I$, where I is some ideal of polynomials vanishing on K_S. In the next section we will be interested in the case where $n = 1$ and $I = \{0\}$. Note: For K_S compact, $\sigma, \tau > 0$ on K_S implies, by [Sc1, Cor. 3], that $\sigma, \tau \in T_S$.

Corollary 2.2 *Let K_S be compact, $f, g \geq 0$ on K_S. Assume that f and g are relatively prime modulo $T_S \cap -T_S$ and that $fg \in T_S$. Then $f \in T_S$ and $g \in T_S$.*

Proof: Apply Lemma 2.1 with $C = K_S$, $A = \mathbb{R}[X]/(T_S \cap -T_S)$, together with [Sc1, Cor. 3], to obtain $\sigma, \tau \in T_S$, $\upsilon \in T_S \cap -T_S$ such that $1 = \sigma f + \tau g + \upsilon$. Then $f = \sigma f^2 + \tau fg + f\upsilon \in T_S$ and $g = \sigma fg + \tau g^2 + g\upsilon \in T_S$. $\square$

Corollary 2.3 *Assume $f, g \in \mathbb{R}[X]$ are relatively prime. Assume further that $K_{\{fg\}}$ is compact and $f, g \geq 0$ on $K_{\{fg\}}$. Then*
(i) There exist σ, $\tau \in \sum \mathbb{R}[X]^2$ such that $1 = \sigma f + \tau g$.
(ii) $M_{\{fg\}} = M_{\{f,g\}} = T_{\{fg\}} = T_{\{f,g\}}$.

See [S1, Prop. 4.8] for another proof of Corollary 2.3 (i) in the case $n = 1$.

Proof: Apply Lemma 2.1 with $C = K_{\{fg\}}$ and $A = \mathbb{R}[X]$ and [Sc1, Cor. 3] to obtain $1 = \sigma f + \tau g$, $\sigma, \tau \in T_{\{fg\}}$. Thus there exist $\sigma_i, \tau_i \in \sum \mathbb{R}[X]^2$, $i = 0, 1$ such that $\sigma = \sigma_0 + \sigma_1 fg$, $\tau = \tau_0 + \tau_1 fg$. Then

$$1 = \sigma f + \tau g = (\sigma_0 + \sigma_1 fg)f + (\tau_0 + \tau_1 fg)g = (\sigma_0 + \tau_1 g^2)f + (\tau_0 + \sigma_1 f^2)g.$$

This proves (i): $1 = \sigma f + \tau g$, $\sigma, \tau \in \sum \mathbb{R}[X]^2$. Thus $fg = \sigma f^2 g + \tau g^2 f \in M_{\{f,g\}}$. Also, $f = \sigma f^2 + \tau fg \in T_{\{fg\}} = M_{\{fg\}}$ and, similarly, $g \in T_{\{fg\}} = M_{\{fg\}}$. This proves (ii). $\square$

Lemma 2.1 is also closely related to results in [S2]. We illustrate by giving a proof of two key results in [S2]:

Corollary 2.4 *Let K_S be compact, $f \geq 0$ on K_S, and suppose*

$$f \in T_S + f\sqrt{(f) + (T_S \cap -T_S)}.$$

Then $f \in T_S$.

Proof: $f = \sigma + fg$, $\sigma \in T_S$, $g^k = af + \tau$, $k \geq 1$, $a \in \mathbb{R}[X]$, $\tau \in T_S \cap -T_S$. $af + (1 - g^k) = 1 - \tau$ so $f, 1 - g$ are relatively prime modulo $T_S \cap -T_S$. Also $f(1 - g) = \sigma \in T_S$. Using this, and the definition of g, one checks that $1 - g \geq 0$ on K_S. Corollary 2.2 yields $f \in T_S$. $\square$

Corollary 2.5 *Let K_S be compact, $f \geq 0$ on K_S and suppose $f = \sigma + \tau b$, where $\sigma, \tau \in T_S$ and b is such that $f = 0 \Rightarrow b > 0$ on K_S. Then $f \in T_S$.*

Proof: By [Sc1, Cor. 3] applied to the set $S \cup \{-f^2\}$, $b = \alpha - \beta f^2$, $\alpha, \beta \in T_S$. Then $f = \sigma + \tau b = \sigma + \tau(\alpha - \beta f^2) = (\sigma + \tau\alpha) - \tau\beta f^2$. The conclusion follows by Corollary 2.4. □

3 Compact Subsets of Lines

In this section, we continue the analysis begun in [K–M, Sect. 2], focusing on the compact case. For related results see [Po–R] and [S2].

If $K \subseteq \mathbb{R}$ is a non-empty closed semi-algebraic set then $K = K_{\mathcal{N}}$, for $\mathcal{N}$ the set of polynomials defined as follows:

— If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in \mathcal{N}$.
— If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in \mathcal{N}$.
— If $a, b \in K$, $a < b$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in \mathcal{N}$.
— $\mathcal{N}$ has no other elements except these.

We call $\mathcal{N}$ **the natural set of generators** for K. We adopt the convention that the natural set of generators of the empty set is $\{-1\}$. The following can be easily deduced from [K–M] and [Sc1]:

Theorem 3.1 *Suppose $n = 1$.*

(a) If K_S is not compact, then T_S is closed, for any finite set of generators S. Moreover the following are equivalent:
(i) T_S is saturated.
(ii) T_S contains the natural set of generators for K_S.
(iii) S contains the natural set of generators of K_S (up to scalings by positive reals).

(b) If K_S is compact, then (†) holds for T_S, for any finite set of generators S. Moreover the following are equivalent:
(i) T_S is saturated.
(ii) T_S is closed.
(iii) T_S contains the natural set of generators for K_S.

Note: Unlike the non-compact case (a), T_S may be saturated in the compact case (b), even when S does *not* contain a scaling of the natural set of generators (cf. [K-M, Notes 2.3(4)]).

Theorem 3.1 (b) is unsatisfactory unless we have a practical criterion on S for determining when T_S is saturated. The following is such a criterion. Here, g' denotes the derivative of g.

Theorem 3.2 *Let* $K_S = \cup_{j=0}^{k}[a_j\ ,\ b_j]$*,* $b_{j-1} < a_j$*,* $j = 1, \ldots, k$*,* $S = \{g_1, \cdots, g_s\}$*. Then* T_S *is saturated if and only if the following two conditions hold:*

(a) for each endpoint a_j $\exists\ i \in \{1, \cdots, s\}$ *such that* $g_i(a_j) = 0$ *and* $g_i'(a_j) > 0$*,*

(b) for each endpoint b_j $\exists\ i \in \{1, \cdots, s\}$ *such that* $g_i(b_j) = 0$ *and* $g_i'(b_j) < 0$*.*

In fact, Theorem 3.2 is just a special case of a general criterion for curves proved in [S2, Th. 5.17]. For completeness we include our own proof.

Proof: We prove the necessity of condition (a). The necessity of condition (b) is proved similarly. Let $a = a_j$. There exists $f \in \mathbb{R}[X]$ (of degree two), $f \geq 0$ on K_S, $f(a) = 0$, $f'(a) > 0$. Since T_S is saturated, f has a presentation $f = \sum \sigma_e g^e$, $\sigma_e \in \sum \mathbb{R}[X]^2$. Since $f(a) = 0$, each term $\sigma_e g^e$ is divisible by $X - a$. Since $f'(a) > 0$, there exists e with $(\sigma_e g^e)'(a) > 0$. If $(X - a)^2 | \sigma_e g^e$ then $(\sigma_e g^e)'(a) = 0$, a contradiction. Thus $\sigma_e(a) \neq 0$ and there is some unique i such that g_i appears in g^e and $g_i(a) = 0$. Then $\sigma_e g^e / g_i$ is strictly positive at a so $(\sigma_e g^e)'(a)$ has the same sign as $g_i'(a)$.

We prove the sufficiency of conditions (a) and (b). Assume (a) and (b) hold. According to Theorem 3.1 (b), it suffices to show that T_S contains the natural generators. We begin with the linear generators. Let $a = a_0$. By (a) there exists i such that $g_i = (X - a)h_i$, $h_i(a) = g_i'(a) > 0$. Then h_i and $X - a$ are each ≥ 0 on K_S and $\gcd(h_i, X - a) = 1$ so, by Corollary 2.2, $X - a \in T_S$. A similar argument shows that $b - X \in T_S$, where $b = b_k$. Now consider a generator of the form $(X - a)(X - b)$, $a = a_\ell$, $b = b_{\ell-1}$, $\ell \in \{1, \ldots, k\}$. By (a) and (b) there exists i, j such that $g_i(a) = 0$, $g_i'(a) > 0$, $g_j(b) = 0$, $g_j'(b) < 0$. Factor g_i, g_j as $g_i = (X - a)h_i$, $g_j = (X - b)h_j$. Then

$$(X - a)(X - b)((X - b)h_i + (X - a)h_j) = (X - b)^2 g_i + (X - a)^2 g_j \in T_S,$$

and $(X - a)(X - b)$ and $(X - b)h_i + (X - a)h_j$ are each ≥ 0 on K_S and are relatively prime, so by Corollary 2.2, $(X - a)(X - b) \in T_S$. □

Corollary 3.3 *Let* K_S *be compact,* $S = \{g_1, \cdots, g_s\}$*. Assume that* K_S *has no isolated points. Then* T_S *is saturated if and only if, for each endpoint* $a \in K_S$*, there exists* $i \in \{1, \cdots, s\}$ *such that* $x - a$ *divides* g_i *but* $(x - a)^2$ *does not.*

Corollary 3.4 *Let* K *be a compact semi-algebraic set in* $\mathbb{R}$ *without isolated points,* $\mathcal{N}$ *its set of natural generators, and* π *the product of the elements of* $\mathcal{N}$*. Then*

$$M_{\{\pi\}} = M_{\mathcal{N}} = T_{\{\pi\}} = T_{\mathcal{N}}.$$

Proof: Applying Corollary 3.3 to $S = \{\pi\}$ we see that $T_{\{\pi\}}$ is saturated. Since $K_{\{\pi\}} = K = K_{\mathcal{N}}$, and $T_{\mathcal{N}}$ is saturated, this implies $T_{\{\pi\}} = T_{\mathcal{N}}$. It remains to show that $\pi \in M_{\mathcal{N}}$. Decompose π as

$$\pi = f q_1 \ldots q_t,$$

where $f = (X-a)(b-X)$ is the product of the linear natural generators and $q_1, \ldots, q_t$ are the quadratic natural generators, $t \geq 0$. Suppose $t = 0$. Then $\pi = (X-a)(b-X) \in M_{\{X-a,b-X\}}$ by Corollary 2.3. Suppose $t > 0$. Then $\pi = \pi' q_t$, where $\pi' = f q_1 \ldots q_{t-1}$. By Corollary 2.3, $\pi \in M_{\{\pi', q_t\}}$. By induction on t, $\pi' \in M_{\mathcal{N}'}$, where $\mathcal{N}' = \{X-a, b-X, q_1, \ldots, q_{t-1}\}$. Thus $\pi \in M_{\mathcal{N}' \cup \{q_t\}} = M_{\mathcal{N}}$. □

We need "no isolated points" in Corollaries 3.3 and 3.4: Take $K = \{0\}$, so $\mathcal{N} = \{X, -X\}$ and $\pi = (X)(-X) = -X^2$. Also $M_{\{-X^2\}} \neq M_{\{X,-X\}}$. On the other hand, writing $-X = a^2 - b^2$, $a, b \in \mathbb{R}[X]$, we see that

$$-X^2 = (a^2 - b^2)X = a^2 X + b^2(-X) \in M_{\{X,-X\}},$$

so $M_{\{X,-X\}} = T_{\{X,-X\}}$. This suggests that $M_{\mathcal{N}} = T_{\mathcal{N}}$ may hold even when K has isolated points. In fact this is the case.

Theorem 3.5 *Let K be a compact semi-algebraic set in $\mathbb{R}$, $\mathcal{N}$ its set of natural generators. Then $M_{\mathcal{N}} = T_{\mathcal{N}}$.*

Note: If K is not compact, then $M_{\mathcal{N}} = T_{\mathcal{N}}$ holds only in a few cases: if $|\mathcal{N}| = 1$ or if $|\mathcal{N}| = 2$ and K has an isolated point; see [K–M, Th. 2.5].

Proof: Let $\mathcal{N} = \{g_1, \ldots, g_s\}$. It suffices to show that $M_{\mathcal{N}}$ is closed under multiplication, equivalently, that $g_i g_j \in M_{\mathcal{N}}$ for all $i \neq j$.

Case 1: g_i, g_j are both linear. $g_i = X-a$, $g_j = b-X$, $a \leq b$. If $a = b$ then $g_j = -g_i$. Write $g_j = f^2 - g^2$, $f, g \in \mathbb{R}[X]$. Then

$$g_i g_j = f^2 g_i - g^2 g_i = f^2 g_i + g^2 g_j \in M_{\{g_i, g_j\}}.$$

If $a < b$ then $\{g_i, g_j\}$ are the natural generators of the compact set $[a, b]$, so $g_i g_j \in M_{\{g_i, g_j\}}$ by Corollary 3.4.

Case 2: g_i is linear, g_j is quadratic. Replacing X by $-X$ if necessary, we can assume $g_i = X - a$. $g_j = (X-c)(X-d)$, $a \leq c < d$. If $a = c$ then $g_i g_j \in M_{\{g_i, g_j\}}$ by [K–M, Th. 2.5]. Suppose $a < c$. Fix $\beta \in \mathbb{R}$ so large that $\beta - X > 0$ on K. $\beta - X \in M_{\mathcal{N}}$ by [J–P, Remark 4.7]. Applying Corollary 3.4 to $[a, c] \cup [d, \beta]$, we see that

$$g_i g_j = (X-a)(X-c)(X-d) \in M_{\{X-a, (X-c)(X-d), \beta - X\}} \subseteq M_{\mathcal{N}}.$$

Case 3: $g_i = (X-c)(X-d)$, $g_j = (X-c')(X-d')$, $c < d \leq c' < d'$. If $c' = d$ then $g_i g_j \in M_{\{g_i, g_j\}}$ by [K–M, Th. 2.5]. Suppose $c' > d$. Use [J–P, Remark 4.7] to choose $\alpha < c$ and $\beta > d'$ so that $X - \alpha, \beta - X \in M_{\mathcal{N}}$. Applying Corollary 3.4 to $[\alpha, c] \cup [d, c'] \cup [d', \beta]$ yields

$$g_i g_j \in M_{\{X-\alpha, (X-c)(X-d), (X-c')(X-d'), \beta - X\}} \subseteq M_{\mathcal{N}}.$$

□

Corollary 3.6 *For $n = 1$, K_S compact, the following are equivalent:*
(i) M_S *is saturated, i.e.,* $M_S = T_S^{\text{alg}}$
(ii) M_S *contains the natural set of generators of* K_S
(iii) M_S *is closed.*

Proof: By [J–P, Remark 4.7] (†) holds for M_S so, in particular, (SMP) holds for M_S. (i) $\Leftrightarrow$ (iii) follows from this fact. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i) follows from Theorem 3.5 and Theorem 3.1(b). □

Corollary 3.6 can be viewed as the module analog of Theorem 3.1 (b). One would also like to have a module analog of Theorem 3.2.

4 Uniform Degree Bounds

We continue to assume $n = 1$. Theorems 4.1 and 4.5 and Corollary 4.3, in this section, are essential in the next section, in the proof of Theorem 5.3 and Corollary 5.5. These results, suitably formulated, hold over an arbitrary real closed field.

Fix a non-empty basic closed semi-algebraic set K in $\mathbb{R}$, with $\mathcal{N} = \{s_1, \dots, s_t\}$ its set of natural generators. $T_{\mathcal{N}}$ is saturated, by Theorem 3.1, so each $P \in \mathbb{R}[X]$ non-negative on K is expressible as

(1) $$P = \sum_{e \in \{0,1\}^t} \sigma_e s_1^{e_1} \dots s_t^{e_t}$$

with $\sigma_e \in \sum \mathbb{R}[X]^2$.

Theorem 4.1 *Each P nonnegative on K has a presentation (1), with the degree of each term $\sigma_e s_1^{e_1} \dots s_t^{e_t}$ bounded by the degree of P.*

Proof: This is just a matter of keeping track of degrees in the proof of [K–M, Th. 2.2]. □

Corollary 4.2 *Suppose $K = K_{\{g_1,\dots,g_k\}}$ and $T_{\{g_1,\dots,g_k\}}$ is saturated. Then each P nonnegative on K has a presentation*

(2) $$P = \sum_{i \in \{0,1\}^k} \tau_i g_1^{i_1} \dots g_k^{i_k},$$

$\tau_i \in \sum \mathbb{R}[X]^2$, where the degrees of the terms are bounded by some number of the form deg(P) + N, with N depending only on $g_1, \dots, g_k$.

Proof: There exist presentations

$$s_j = \sum_{p\in\{0,1\}^k} \rho_{jp} g_1^{p_1}\dots g_k^{p_k},$$

with $\rho_{jp} \in \sum \mathbb{R}[X]^2$, for each j. Substituting these into (1) and rearranging terms, one obtains (2), with the degree of each term at most

$$\deg(P) + t\max\{\deg(\rho_{jp} g_1^{p_1}\dots g_k^{p_k}) : j, p\}.$$

□

It does not seem possible to choose N in Corollary 4.2 so as to depend only on the *degrees* of $g_1,\dots,g_k$. Unfortunately, this limits what we are able to prove later, in Section 5, in Theorem 5.3.

We consider module analogs of Theorem 4.1. We continue to denote by $s_1,\dots,s_t$ the natural generators of K. Is it possible to find presentations

$$P = \sigma_0 + \sigma_1 s_1 + \dots + \sigma_t s_t,$$

with $\sigma_i \in \sum \mathbb{R}[X]^2$, with the terms of bounded degree depending only on the degree of P? For $t \le 1$ it is trivially true, by Theorem 4.1, (with $\deg(P)$ serving as the bound). If K is not compact and $t \ge 2$, then such a presentation may not exist. ($M_{\mathcal{N}}$ is saturated iff either $t \le 1$ or if $t = 2$ and K has an isolated point; see [K–M, Th. 2.5].) If such a presentation does exist, then $\deg(P)$ serves as a degree bound. If K is compact and $t = 2$, then one checks that $\deg(P) + 1$ serves as a degree bound. To summarize:

Corollary 4.3 *If either $t \le 1$ or $t = 2$ and K either has an isolated point or is compact, then every polynomial P non-negative on K has a presentation*

$$P = \sigma_0 + \sigma_1 s_1 + \dots + \sigma_t s_t,$$

with the degree of each term bounded by deg$(P) + 1$.

What if K is compact and $t \ge 3$? In view of Theorem 4.1, this reduces to the question of whether a degree bound exists for representations of the products $s_i s_j$, $i < j$. Such a bound exists if s_i, s_j are both linear, or, if at least one of s_i, s_j is quadratic and $K_{\{s_i,s_j\}}$ has an isolated point. In remaining cases (i.e, the cases where at least one of s_i, s_j is quadratic and $K_{\{s_i,s_j\}}$ has no isolated point), there is no degree bound in general.

Example 4.4 (a) Suppose $s_1 = X - a$, $s_2 = (X - b)(X - c)$, where $a < b < c$ are fixed reals. Then, for any real $\beta > c$, $s_1 s_2$ has a presentation

(3), $$s_1 s_2 = \sigma_0 + \sigma_1 s_1 + \sigma_2 s_2 + \sigma_3(\beta - X)$$

with $\sigma_i \in \sum \mathbb{R}[X]^2$ depending on β. We *claim* that, for any such presentation, as $\beta \to \infty$, the maximum of the degrees of the σ_i necessarily tends to infinity. Otherwise, by the Transfer Principle, for any real closed field $R \supseteq \mathbb{R}$ and any $\beta \in R$, $\beta > c$, we would have a presentation (3) with $\sigma_i \in \sum R[X]^2$. Choose any such (nonarchimedean) R, e.g., take R to be the field of formal Puiseux series with real coefficients. Denote by v the unique finest valuation on R compatible with the ordering, and choose β to be positive and infinitely large relative to $\mathbb{R}$, i.e., $v(\beta) < 0$. Write $\sigma_i = \sum f_{ij}^2$, $f_{ij} \in R[X]$. Choose $d \in R$, $d > 0$, with $v(d)$ equal to the minimum of the values of the coefficients of the f_{ij}, $i \in \{0, 1, 2\}$ and the $\sqrt{\beta} f_{3j}$.

Case 1: $v(d) \geq 0$. Then (3) pushes down to the residue field giving an equation of the form $s_1 s_2 = \alpha_0 + \alpha_1 s_1 + \alpha_2 s_2$, $\alpha_i \in \sum \mathbb{R}[X]^2$. This contradicts [K-M, Th. 2.5].

Case 2: $v(d) < 0$. Then dividing (3) by d^2 and pushing down to the residue field yields an equation of the form $0 = \alpha_0 + \alpha_1 s_1 + \alpha_2 s_2$, $\alpha_i \in \sum \mathbb{R}[X]^2$, $\alpha_i \neq 0$ for some i. Since s_1 and s_2 are both strictly positive on the infinite set (a, b), this is not possible.

(b) A similar remark applies to presentations of the form

$$s_1 s_2 = \sigma_0 + \sigma_1 s_1 + \sigma_2 s_2 + \sigma_3(\beta + X) + \sigma_4(\beta - X), \ \ \sigma_i \in \mathbb{R}[X]^2,$$

where $s_1 = (X - a)(X - b)$, $s_2 = (X - c)(X - d)$, $a < b < c < d$ are fixed reals, $\beta > \max\{-a, d\}$.

We turn now to the case where K is the empty set. We use the next result in the proof of both Theorem 5.3 and Corollary 5.5. Actually, for the proof of Theorem 5.3, we need only a weaker preordering version of the result. The weaker preordering version holds even in the multivariable situation, and is a consequence of the Positivstellensatz.

Theorem 4.5 *Given a positive integer k and non-negative integers $d_1, \ldots, d_k$, there exists a positive integer N such that for any real closed field R and any set of polynomials $S = \{g_1, \ldots, g_k\}$ in one variable X with coefficients in R with $deg(g_j) \leq d_j$, $j = 1, \ldots, k$, if $K_S = \emptyset$, then there exist $\sigma_i \in \sum R[X]^2$ of degree $\leq N$ such that $-1 = \sum_{i=0}^k \sigma_i g_i$. (Convention: $g_0 = 1$.)*

Proof: If $-1 \notin M_S$ then, by [Br, Satz 1.8], there exists a real prime ideal p in $R[X]$ such that the quadratic form ϕ^*, which is obtained from $\phi = \langle 1, g_1, \ldots, g_k \rangle$ by deleting the entries belonging to p, is strongly anisotropic over the residue field, call it $k(p)$, of $R[X]$ at p. (This works even in the multivariable situation, in fact it works for any commutative ring.) The point is, since we are in the 1-variable situation, $k(p)$ has transcendence degree ≤ 1 over R, so there exists an ordering of $k(p)$ making ϕ^* positive definite [P, Th. 9.4]. By the Transfer Principle, this implies $K_S \neq \emptyset$. The degree bounds follow by a standard ultrapower argument. □

Note: Using the identity $P = (\frac{P+1)}{2})^2 - (\frac{P-1}{2})^2$, we also get degree bounds for presentations $P = \sum_{i=0}^{k} \sigma_i g_i$ depending only on k and the degree of P and the degrees of $g_1, \ldots, g_k$, for each $P \in R[X]$.

5 Subsets of Cylinders

We fix the terminology. We consider the polynomial ring $\mathbb{R}[X, Y]$ in $n+1$ variables $X_1, \ldots, X_n, Y$ with coefficients in $\mathbb{R}$. We suppose S is a finite subset of $\mathbb{R}[X, Y]$. When is it true that (SMP) (or (‡) or (†)) holds for T_S?

Our first result gives a general necessary condition. Let $L \subseteq \mathbb{R}^{n+1}$ be any line,

$$\lambda : \mathbb{R} \xrightarrow{\cong} L \subseteq \mathbb{R}^{n+1},$$

an affine-linear isomorphism onto this line, and $\lambda^* : \mathbb{R}[X, Y] \to \mathbb{R}[Z]$ the corresponding map between the polynomial rings. Thus $\lambda^{-1}(K_S)$ is the basic closed semi-algebraic subset of $\mathbb{R}$ defined by the set $\lambda^*(S)$.

Theorem 5.1 *Suppose that (SMP) holds for T_S and $\lambda^{-1}(K_S)$ is not compact. Then $\lambda^*(S)$ contains the natural generators of $\lambda^{-1}(K_S)$ (up to scaling by positive reals).*

Note: If $\lambda^{-1}(K_S)$ is compact then $T_{\lambda^*(S)}$ satisfies (†). Thus, one can conclude that $T_{\lambda^*(S)}$ satisfies (†) in any case.

Proof: By Theorem 3.1, it suffices to prove that the (closed) preordering $T_{\lambda^*(S)}$ contains the natural generators of $\lambda^{-1}(K_S)$. To simplify the notation needed in the proof, the coordinates of $\mathbb{R}^{n+1}$ are arranged so that L is the Y-axis and $\lambda : \mathbb{R} \to L \subseteq \mathbb{R}^{n+1}$ is the isomorphism defined by $y \to (0, y)$. Thus, $\lambda^* : \mathbb{R}[X, Y] \to \mathbb{R}[Y]$ is given by $X_i \mapsto 0$, $i = 1, \ldots, n$, $Y \mapsto Y$. If $\lambda^{-1}(K_S) = \mathbb{R}$ there is nothing to prove. So assume that $\lambda^{-1}(K_S) \neq \mathbb{R}$.

Case 1: Suppose that $\lambda^{-1}(K_S)$ contains a smallest element, say a. (The case of a largest element is handled in exactly the same way.) The natural generator belonging to a is the linear polynomial $Y - a$. It is claimed that $Y - a$ belongs to $T_{\lambda^{-1}(S)}$. For each $1 \leq n \in \mathbb{N}$ define $b_n = a - \frac{1}{n}$, $c_n = b_n - n$. Then: $(b_n)_n$ increases monotonically with limit a; $(c_n)_n$ decreases monotonically with limit $-\infty$; for all n, $b_n > c_n$. The interval $[c_n, b_n] \subseteq L$ is compact and does not meet the closed set K_S. Thus, the distance $\delta_n = \text{dist}([c_n, b_n], K_S)$ is positive.

For each n, let E_n be the ellipsoid with center $\frac{1}{2}(b_n + c_n)$, one axis on the line L, the other perpendicular to L. The half axis on L has length $\frac{1}{2}(b_n - c_n)$, the perpendicular half axis has length δ_n. The ellipsoid is given by the polynomial

$$P_n = \left(\frac{Y - \frac{1}{2}(b_n + c_n)}{\frac{1}{2}(b_n - c_n)} \right)^2 + \sum_{i=1}^{n} \left(\frac{X_i}{\delta_n} \right)^2 - 1.$$

The interior of the ellipsoid is the set

$$N(P_n) = \{(x,y) \in \mathbb{R}^{n+1} : P_n(x,y) < 0\}.$$

Every point of $N(P_n)$ has a distance less than δ_n from the line segment $[c_n, b_n]$, hence $N(P_n) \cap K_S = \emptyset$. This means that P_n is positive semidefinite on K_S. Each linear functional L_0 on $\mathbb{R}[Y]$ non-negative on $T_{\lambda^*(S)}$ lifts to a linear functional $L_1 = L_0 \circ \lambda^*$ on $\mathbb{R}[X,Y]$ non-negative on T_S. Since T_S satisfies (SMP), $L_0(\lambda^*(P_n)) = L_1(P_n) \geq 0$ for all such L_0. It follows that $\lambda^*(P_n) \in T^{\text{lin}}_{\lambda^*(S)} = T_{\lambda^*(S)}$. One computes

$$\lambda^*(P_n) = \frac{4}{b_n - c_n} \cdot \left(\frac{1}{b_n - c_n} Y^2 + \left(1 - \frac{2b_n}{b_n - c_n}\right) Y - \left(b_n - \frac{b_n^2}{b_n - c_n}\right)\right)$$

Since $\frac{1}{b_n - c_n}$ converges to 0 as $n \to \infty$ one concludes that the sequence

$$\left(\frac{b_n - c_n}{4} \lambda^*(P_n)\right)_n$$

of polynomials converges to $Y - a$. Once again, closedness of $T_{\lambda^*(S)}$ implies $Y - a \in T_{\lambda^*(S)}$.

Case 2: Suppose that $a, b \in \lambda^{-1}(K_S), a < b$, $(a,b) \cap \lambda^{-1}(K_S) = \emptyset$. It is claimed that the natural generator $(Y-a)(Y-b)$ belongs to $T_{\lambda^*(S)}$. The method of proof is much the same as in Case 1; let $n_0 \in \mathbb{N}$ be such that $b - a > \frac{2}{n_0}$. For $n \geq n_0$ one defines $a_n = a + \frac{1}{n}$, $b_n = b - \frac{1}{n}$. Then: $a_n < b_n$; $a_n \downarrow a$; $b_n \uparrow b$; the compact line segment $[a_n, b_n]$ does not meet K_S. Set $\delta_n = \text{dist}([a_n, b_n], K_S)$, and let

$$P_n = \left(\frac{Y - \frac{1}{2}(b_n + a_n)}{\frac{1}{2}(b_n - a_n)}\right)^2 + \sum_{i=1}^n \left(\frac{X_i}{\delta_n}\right)^2 - 1.$$

Again, $N(P_n)$ is the interior of the associated ellipsoid E_n, and $N(P_n) \cap K_S = \emptyset$. Thus, $P_n \geq 0$ on K_S. Again, since T_S satisfies (SMP) and $T_{\lambda^*(S)}$ is closed one gets $\lambda^*(P_n) \in T_{\lambda^*(S)}$. The sequence

$$\left(\left(\frac{1}{2}(b_n - a_n)\right)^2 \cdot \lambda^*(P_n)\right)_n$$

converges to

$$\left(Y - \tfrac{1}{2}(b+a)\right)^2 - \left(\tfrac{1}{2}(b-a)\right)^2$$

$$= Y^2 - (b+a)Y + ab = (Y-a)(Y-b).$$

Once again, since $T_{\lambda^*(S)}$ is closed the limit also belongs to $T_{\lambda^*(S)}$. □

For each $a \in \mathbb{R}^n$, let $L_a = \lambda_a(\mathbb{R})$, where $\lambda_a : \mathbb{R} \to \mathbb{R}^{n+1}$ is defined by $y \mapsto (a, y)$. The corresponding map $\lambda_a^* : \mathbb{R}[X,Y] \to \mathbb{R}[Y]$ is given by $X_i \mapsto a_i, Y \mapsto Y$. The fiber $\lambda_a^{-1}(K_S)$ is the basic closed set in $\mathbb{R}$ defined by $\lambda_a^*(S)$.

Theorem 5.1 and [Sc2, Th. 1] (see Theorem 1.5) combine to yield the following:

Corollary 5.2 *Let $S \subseteq \mathbb{R}[X, Y]$, where $\mathbb{R}[X, Y]$ denotes the polynomial ring in $n+1$ variables $X_1, \ldots, X_n, Y$. Suppose the image of K_S under the projection $(x, y) \mapsto x$ is bounded. For each $a \in \mathbb{R}^n$, define $\lambda_a : \mathbb{R} \to \mathbb{R}^{n+1}$ as above. Then:*

(a) T_S satisfies (MP).

(b) The following are equivalent:

(i) T_S satisfies (SMP).

(ii) For each non-compact $\lambda_a^{-1}(K_S)$, $\lambda_a^*(S)$ contains the natural generators for $\lambda_a^{-1}(K_S)$ (up to scalings by positive reals).

Proof: (a) Since every polynomial non-negative on $\mathbb{R}$ is a sum of squares, (MP) holds for each $T_{\lambda_a^*(S)}$, so (a) is immediate from [Sc2, Th. 1]. (b) The implication (i) $\Rightarrow$ (ii) follows from Theorem 5.1. (ii) $\Rightarrow$ (i): To be able to apply [Sc2, Th. 1] we must show that each $T_{\lambda_a^*(S)}$ satisfies (SMP). For $\lambda_a^{-1}(K_S)$ compact, this is always the case. For $\lambda_a^{-1}(K_S)$ non-compact it is a consequence of assumption (ii). $\square$

See Corollary 6.4 below for a module version of Corollary 5.2.

We are unable to prove that Corollary 5.2(b) holds with (SMP) replaced by (‡). However, we are able to prove the following weaker result:

Theorem 5.3 *Let $S \subseteq \mathbb{R}[X, Y]$, where $\mathbb{R}[X, Y]$ denotes the polynomial ring in $n+1$ variables $X_1, \ldots, X_n, Y$. Suppose the following conditions hold:*

(a) There exists N such that $N - \|X\|^2 \in T_S$ and

(b) For each non-empty $\lambda_a^{-1}(K_S)$, $\lambda_a^*(S)$ contains the natural generators for $\lambda_a^{-1}(K_S)$ (up to scaling by positive reals).

Then T_S satisfies (‡), i.e., for each $P \in \mathbb{R}[X, Y]$ such that $P \geq 0$ on K_S, there exists $Q \in \mathbb{R}[X, Y]$ such that for all real $\epsilon > 0$, $P + \epsilon Q \in T_S$.

See Corollary 5.5 below for a module version of Theorem 5.3.

Note: It follows from [Sc1, Cor. 3] that condition (a) of Theorem 5.3 holds iff there exist elements $f_1, \ldots, f_t \in T_S \cap \mathbb{R}[X]$ such that the subset $K_{\{f_1,\ldots,f_t\}}$ of $\mathbb{R}^n$ is compact.

Proof: Let $B = \{a \in \mathbb{R}^n : \|a\|^2 \leq N\}$. Suppose that $P \in \mathbb{R}[X, Y]$ belongs to T_S^{alg}. Writing $P = \sum_{i=0}^k p_i(X) Y^i$ one has $P_a := \lambda_a^*(P) = \sum_{i=0}^k p_i(a) Y^i \in T^{\text{alg}}_{\lambda_a^*(S)}$ for each $a \in B$. Because of condition (b), $T_{\lambda_a^*(S)} = T^{\text{alg}}_{\lambda_a^*(S)}$ for each a. Thus, for each a there is a presentation

$$(4) \qquad P_a = \sum_{i \in \{0,1\}^t} \sigma_{a,i} \lambda_a^*(s_1)^{i_1} \cdot \ldots \cdot \lambda_a^*(s_t)^{i_t}$$

(where $S = \{s_1, \ldots, s_t\}$ and the $\sigma_{a,i} \in \mathbb{R}[Y]$ are sums of squares). By assumption (b) we may assume there is a uniform bound for the degrees of the sums of squares occuring in these representations that is independent from a. If $\lambda_a^{-1}(K_S) \neq \emptyset$ then the degrees can be bounded by k by Theorem 4.1. For $\lambda_a^{-1}(K_S) = \emptyset$ a uniform bound exists by the note following Theorem 4.5. Pick any $\varepsilon > 0$. For each a, consider

$$\sum_{i \in \{0,1\}^t} \sigma_{a,i} s_1^{i_1} \cdot \ldots \cdot s_t^{i_t}, \quad P - \sum_{i \in \{0,1\}^t} \sigma_{a,i} s_1^{i_1} \cdot \ldots \cdot s_t^{i_t}$$

as polynomials in the variable Y with coefficients from $\mathbb{R}[X]$. There is some large even number D that bounds the Y–degrees of these polynomials and is independent from a. Write

$$\sum_{i \in \{0,1\}^t} \sigma_{a,i} s_1^{i_1} \cdot \ldots \cdot s_t^{i_t} = \sum_{d=0}^{D} p_{a,d} Y^d, \quad P - \sum_{i \in \{0,1\}^t} \sigma_{a,i} s_1^{i_1} \cdot \ldots \cdot s_t^{i_t} = \sum_{d=0}^{D} q_{a,d} Y^d.$$

Note that $q_{a,0}(a) = \ldots = q_{a,D}(a) = 0$ for each $a \in B$. Hence there is an open semi-algebraic neighborhood U_a of a in $\mathbb{R}^n$ such that

$$\sum_{d=0}^{D} \left| q_{a,d}(x) \right| < \frac{\varepsilon}{2}$$

for every $x \in U_a$. By compactness of B, there is a finite subset E of B such that $B \subseteq \bigcup_{a \in E} U_a$. On B there is a continuous semi-algebraic partition of unity subordinate to this cover, say e_a, $a \in E$, with $0 \leq e_a \leq 1$ and $\sum\limits_{a \in E} e_a(x) = 1$ for each $x \in B$. Each e_a has a nonnegative square root f_a, which is also a continuous semi-algebraic function. Using Stone–Weierstrass approximation on the compact set B one finds polynomials F_a, $a \in E$, so close to the continuous semi-algebraic functions f_a that

$$\Big| \sum_{a \in E} p_{a,d}(f_a^2 - F_a^2)(x) \Big| < \frac{\varepsilon}{2}$$

for every $d = 0, \ldots, D$ and every $x \in B$. The polynomial P can be written in the form $P = P_1 + P_2$ where

$$\begin{aligned} P_1 &= \sum_{i \in \{0,1\}^t} \left(\sum_{a \in E} \sigma_{a,i} F_a^2 \right) s_1^{i_1} \cdot \ldots \cdot s_t^{i_t} \in T_S, \\ P_2 &= \sum_{d=0}^{D} r_d Y^d = \sum_{d=0}^{D} \left(\sum_{a \in E} q_{a,d} F_a^2 \right) Y^d \\ &\quad + \sum_{d=0}^{D} \left(\sum_{a \in E} q_{a,d}(f_a^2 - F_a^2) \right) Y^d + \sum_{d=0}^{D} \left(\sum_{a \in E} p_{a,d}(f_a^2 - F_a^2) \right) Y^d. \end{aligned}$$

The choice of the approximations yields

$$\begin{aligned}
\left|r_d(x)\right| &= \Big|\sum_{a\in E} q_{a,d} f_a^2(x) + \sum_{a\in E} p_{a,d}(f_a^2 - F_a^2)(x)\Big| \\
&\leq \Big|\sum_{a\in E} q_{a,d} f_a^2(x)\Big| + \Big|\sum_{a\in E} p_{a,d}(f_a^2 - F_a^2)(x)\Big| \\
&\leq \Big|\sum_{a\in E} q_{a,d}(x)\Big| + \Big|\sum_{a\in E} p_{a,d}(f_a^2 - F_a^2)(x)\Big| < \varepsilon
\end{aligned}$$

for each $x \in B$. But then [Sc1, Cor. 3] implies that $\varepsilon \pm r_d \in T_{\{N-\|X\|^2\}}$ for every $d = 0, \ldots, D$. The argument in [K–M, Th. 5.1] shows that

$$\varepsilon Y^2 + (\varepsilon + r_d)Y + \varepsilon \in (Y+1)^2 T_{\{N-\|X\|^2\}} + (Y^2+1)T_{\{N-\|X\|^2\}}.$$

As in loc.cit., one adds εQ where

$$Q = 2 + Y + 3Y^2 + Y^3 + 3Y^4 + \ldots + 3Y^{D-2} + Y^{D-1} + 2Y^D$$

to both sides of $P = P_1 + P_2$ and obtains

$$\begin{aligned}
P + \varepsilon Q = \quad & \sum_{i=\{0,1\}^t} \left(\sum_{a\in E} \sigma_{a,i} F_a^2\right) s_1^{i_1} \cdot \ldots \cdot s_t^{i_t} \\
& + \sum_{\substack{d=0 \\ d \text{ even}}}^{D} (\varepsilon + r_d)Y^d + \sum_{\substack{d=0 \\ d \text{ odd}}}^{D} \varepsilon Y^{d+1} + (\varepsilon + r_d)Y^d + \varepsilon Y^{d-1} \in T_S.
\end{aligned}$$

Since the polynomial Q is independent from ε, the proof is finished. □

There is one case for which the results proved so far give even a necessary and sufficient condition for (‡) to hold:

Corollary 5.4 *Let $S \subseteq \mathbb{R}[X,Y]$, where $\mathbb{R}[X,Y]$ denotes the polynomial ring in $n+1$ variables $X_1, \ldots, X_n, Y$. Suppose the following conditions hold:*
(a) There exists N such that $N - \|X\|^2 \in T_S$ and
(b) Each nonempty $\lambda_a^{-1}(K_S)$ is unbounded.
Then the following conditions are equivalent:
(i) T_S satisfies condition (‡).
(ii) For each non-empty $\lambda_a^{-1}(K_S)$, $\lambda_a^*(S)$ contains the natural generators for $\lambda_a^{-1}(K_S)$ (up to scaling by positive reals).

If we look carefully at the proof of Theorem 5.3, we see that in fact we have proved more than is stated: We have shown that, for each $\epsilon > 0$, $P = P_1 + P_2$,

$$P_1 \in T_S \quad \text{and} \quad P_2 + \epsilon Q \in \sum_k T_{\{N-\|X\|^2\}} Y^{2k} + \sum_k T_{\{N-\|X\|^2\}} Y^{2k}(1+Y)^2.$$

We may not need all the various sorts of terms in T_S to represent P_1. If $S = \{s_1, \dots, s_k\}$ and $e_i \in \{0,1\}$, $\sum e_i > 1$, we need the term involving $s_1^{e_1} \dots s_k^{e_k}$ only if there exists a such that the $\lambda_a^*(s_i)$ with $e_i = 1$ actually occur as (distinct) natural generators of $\lambda_a^{-1}(K_S)$. For example, if the hypothesis of Corollary 4.3 holds for each non-empty fiber, then we can arrange things so $P_1 \in M_S$.

Corollary 5.5 *Let $S \subseteq \mathbb{R}[X, Y]$, where $\mathbb{R}[X, Y]$ denotes the polynomial ring in $n+1$ variables $X_1, \dots, X_n, Y$. Suppose the following conditions hold:*

(a) There exists N such that $N - \|X\|^2 \in M_S$.

(b) For each non-empty $\lambda_a^{-1}(K_S)$, $\lambda_a^*(S)$ contains the natural generators for $\lambda_a^{-1}(K_S)$ (up to scaling by positive reals).

(c) Each non-empty $\lambda_a^{-1}(K_S)$ has the form $(-\infty, \infty)$ or $(-\infty, p]$, $[q, \infty)$, $(-\infty, p] \cup [q, \infty)$ (perhaps with a discrete point added) or $[p, q]$.

Then M_S satisfies (‡), i.e., for each $P \in \mathbb{R}[X, Y]$ such that $P \geq 0$ on K_S, there exists $Q \in \mathbb{R}[X, Y]$ such that for all real $\epsilon > 0$, $P + \epsilon Q \in M_S$.

6 Module Version of Schmüdgen's Theorem

We prove the following module version of Schmüdgen's Theorem 1.5 quoted in the introduction:

Theorem 6.1 *Let $h_1, \dots, h_d \in \mathbb{R}[X]$. Suppose there exists N such that $N - \sum_{i=1}^d h_i^2 \in M_S$. Denote by $V_h \subseteq \mathbb{R}^d$ the algebraic set associated to the $\mathbb{R}$-algebra $\mathbb{R}[h] := \mathbb{R}[h_1, \dots, h_d]$. Suppose that the image of K_S under the map $\mathbb{R}^n \to V_h$, $x \mapsto (h_1(x), \dots, h_d(x))$, is dense in the set $\{z \in V_h : g(z) \geq 0$ for all $g(h) \in \mathbb{R}[h] \cap M_S\}$. If (SMP) (resp., (MP)) holds for M_{S_λ} for each λ in the closed ball of radius $\sqrt{N}$ about the origin in $\mathbb{R}^d$, then (SMP) (resp., (MP)) holds for M_S.*

The density assumption in Theorem 6.1 is somewhat restrictive. At the same time, this assumption holds in a variety of interesting cases: Consider the case $d = 1$. Suppose $h \in \mathbb{R}[X]$ is non-constant and bounded on K_S. Then V_h is identified with $\mathbb{R}$ and the closure of the image of K_S in V_h is some disjoint union of closed intervals, say $\cup_{i=1}^k [a_i, b_i]$, $b_{i-1} < a_i$. Adding the natural generators $b_k - h$, $h - a_1$, $(h - b_{i-1})(h - a_i)$, $i = 2, \dots, k$ to S does not change K_S. Once this is done, $N - h^2 \in M_S$ for N sufficiently large, and the density assumption of Theorem 6.1 holds. Similar remarks apply if $d > 1$ and $h_1, \dots, h_d \in \mathbb{R}[X]$ are bounded on K_S, provided the closure of the image of K_S in V_h is a basic closed semi-algebraic set in V_h (this latter condition is automatic if $d = 1$).

The proof of Theorem 6.1 follows along the same lines as the proof given in [Sc2]. We need two lemmas.

Lemma 6.2 *Suppose $p \in \mathbb{R}[h]$, $p \geq 0$ on K_S. Then $p + \epsilon \in M_S$ holds for all real $\epsilon > 0$.*

Proof: The hypothesis $N - \sum_{i=1}^{d} h_i^2 \in M_S$ implies the quadratic module $\mathbb{R}[h] \cap M_S$ is Archimedean in $\mathbb{R}[h]$. Let $z \in V_h$ be such that $g(z) \geq 0$ for all $g(h) \in \mathbb{R}[h] \cap M_S$. Write $p = f(h)$, $f \in \mathbb{R}[Z_1, \ldots, Z_d]$. By hypothesis, there is a sequence $x^{(j)}$ in K_S, $(h_1(x^{(j)}), \ldots, h_d(x^{(j)}) \to z$, so $p(x^{(j)}) = f(h_1(x^{(j)}), \ldots, h_d(x^{(j)})) \to f(z)$ as $j \to \infty$. It follows that $f(z) \geq 0$ for any such z. By Jacobi's generalization of the Kadison-Dubois Theorem [J, Th. 4], this implies $p + \epsilon = f(h) + \epsilon \in M_S$ for any real $\epsilon > 0$. $\square$

We use the notation of [Sc2]: L is a linear functional on $\mathbb{C}[X]$, $L(M_S) \geq 0$. Denote by $\mathcal{N}$ the ideal in $\mathbb{C}[X]$ defined by $\mathcal{N} = \{p \in \mathbb{C}[X] : L(p\overline{p}) = 0\}$. The algebra $\mathcal{D}_L = \mathbb{C}[X]/\mathcal{N}$ comes equipped with an involution $p + \mathcal{N} \mapsto \overline{p} + \mathcal{N}$ and a scalar product $\langle p_1 + \mathcal{N}, p_2 + \mathcal{N} \rangle := L(p_1 \overline{p}_2)$. Each $p \in \mathbb{C}[X]$ defines a linear operator $\pi_L(p)$ on $\mathcal{D}_L$ by $\pi_L(p)(q + \mathcal{N}) = pq + \mathcal{N}$. We use the following easy version of [Sc2, Prop. 2]:

Lemma 6.3 *If $p \in \mathbb{C}[h]$ then the operator $\pi_L(p)$ on $\mathcal{D}_L$ is bounded and $\|\pi_L(p)\| \leq \|p\|_K$ where $\|p\|_K := \sup\{|p(x)| : x \in K_S\}$.*

Proof: Using $\|\pi_L(p)\|^2 = \|\pi_L(p\overline{p})\|$ and $\|p\|^2 = \|p\overline{p}\|$, we are reduced to the case $p \in \mathbb{R}[h]$. Fix $\epsilon > 0$ and let $\rho^2 = \|p\|_K^2 + \epsilon$. By Lemma 6.2, $\rho^2 - p^2 \in M_S$ so, for $q \in \mathbb{C}[X]$, $\rho^2 q\overline{q} - p^2 q\overline{q} = (\rho^2 - p^2)q\overline{q} \in M_S$. It follows that $\rho^2 L(q\overline{q}) \geq L(p^2 q\overline{q})$, i.e., $\rho^2 \|q + \mathcal{N}\|^2 \geq \|\pi_L(p)(q + \mathcal{N})\|^2$. Since $q \in \mathbb{C}[X]$ is arbitrary, this implies $\rho \geq \|\pi_L(p)\|$. Since $\epsilon > 0$ is arbitrary, the result follows. $\square$

It follows from Lemma 6.3 that the operators $\pi_L(h_1), \ldots, \pi_L(h_d)$ are bounded. One finishes the proof of Theorem 6.1 now, exactly as in [Sc2].

In terms of basic closed semi-algebraic subsets of cylinders with compact cross-section, set-up as in Section 5, Theorem 6.1 yields the following:

Corollary 6.4 *Let $S \subseteq \mathbb{R}[X, Y]$, where $\mathbb{R}[X, Y]$ denotes the polynomial ring in $n + 1$ variables $X_1, \ldots, X_n, Y$. Suppose $N - \|X\|^2 \in M_S$ and, for each $x \in \mathbb{R}^n$, $x \notin \overline{\pi(K_S)}$, there exists $g \in M_S \cap \mathbb{R}[X]$ such that $g(x) < 0$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection $(x, y) \mapsto x$. Then:*

(a) (MP) holds for M_S.

(b) If (SMP) holds for $M_{\lambda_a^*(S)}$ for each non-compact fiber $\lambda_a^{-1}(K_S)$ then (SMP) holds for M_S.

7 Applications

In [K–M] the authors asked whether [K–M, Cor. 5.4] generalizes to sets of the form $K \times L$, K compact, L non-compact closed semi-algebraic in $\mathbb{R}$ (with the natural description). We are now able to prove this. See [Po, Cor. 3] for another proof.

Denote by $\mathbb{R}[X, Y]$ the polynomial ring in $n+1$ variables $X_1, \dots, X_n, Y$. Consider a subset $S = \{g_1, \dots, g_s\}$ of $\mathbb{R}[X, Y]$ where the polynomials $g_1, \dots, g_s$ involve only the variables $X_1, \dots, X_n$, so K_S has the form $K \times \mathbb{R}$, $K \subseteq \mathbb{R}^n$. We further assume that K is compact. Given L closed semi-algebraic in $\mathbb{R}$, let $\mathcal{N} \subset \mathbb{R}[Y]$ be the natural set of generators for L. In this situation we have the following:

Corollary 7.1 *Let $S_L := S \cup \mathcal{N}$ (so $K_{S_L} = K \times L$). Then (‡) holds for T_{S_L}.*

Proof: Immediate from Theorem 5.3. □

Our next application is to generalized polyhedra. Assume that K_S is the basic closed semi-algebraic set in $\mathbb{R}^m$, $m \geq 1$, defined by $S = \{\ell_1, \dots, \ell_s\}$, where $\ell_1, \dots, \ell_s$ are linear, so K_S is a **closed polyhedron**. If K_S is compact then, by [J–P, Th. 4.2], (†) holds for M_S. What if K_S is not compact? If K_S contains a cone of dimension 2 then, by Theorem 1.2, (MP) fails for T_S. In [K–M] the authors ask whether (‡) holds in the remaining case, i.e., when K_S is not compact and does not contain a cone of dimension 2. We are now in a position to settle this question completely.

Lemma 7.2 *Let P be a closed polyhedron in $\mathbb{R}^m$ and let $a \in P$. Then P is compact if and only if there is no half line starting at a and contained in P.*

Lemma 7.3 *Let P be a non-compact closed polyhedron in $\mathbb{R}^m$ and let $a \in P$. Then the following are equivalent:*

(i) There are at most two half lines starting at a and contained in P.

(ii) There is a subset S_1 of the set of linear polynomials defining P such that K_{S_1} is a cylinder with compact cross-section.

The proofs of Lemmas 7.2 and 7.3 are elementary and will be omitted.

Theorem 7.4 *Let P be a closed polyhedron in $\mathbb{R}^m$ defined by a finite set S of linear polynomials.*

(i) If P is compact then (†) holds for M_S.

(ii) If P is not compact but does not contain a 2-dimensional cone then (‡) holds for M_S.

(iii) If P contains a 2-dimensional cone then (MP) fails for T_S.

Proof: It only remains to prove (ii). We can assume $m \geq 2$. Set $n = m - 1$. According to Lemmas 7.2 and 7.3, after a suitable affine change in coordinates, there exists a subset $S_1 \subseteq S \cap \mathbb{R}[X_1, \ldots, X_n]$ such that K_{S_1} is a cylinder with compact cross-section. The result follows, applying Corollary 5.5. The fact that condition (a) of Corollary 5.5 holds follows by applying [J–P, Theorem 4.2] to the compact polyhedron $K_{S_1} \cap \mathbb{R}^n$. □

8 Examples in the Plane

Example 8.1 Consider $\{Y - X^2\} \subseteq \mathbb{R}[X, Y]$. $K_{\{Y-X^2\}}$ is the region above the parabola $Y = X^2$ in $\mathbb{R}^2$. $K_{\{Y-X^2\}}$ does not contain a 2–dimensional cone. On the other hand, the isomorphism

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x, y - x^2),$$

carries $K_{\{Y-X^2\}}$ onto $K_{\{Y\}}$ which does contain a 2–dimensional cone. By [K–M, Th. 3.5] and Proposition 1.2, $T_{\{Y\}} = T^{\text{lin}}_{\{Y\}} \not\supseteq T^{\text{alg}}_{\emptyset}$. Applying the induced algebra isomorphism

$$\phi^* : \mathbb{R}[X, Y] \to \mathbb{R}[X, Y], \quad X \mapsto X, \quad Y \mapsto Y - X^2,$$

yields $T_{\{Y-X^2\}} = T^{\text{lin}}_{\{Y-X^2\}} \not\supseteq T^{\text{alg}}_{\emptyset}$, so $T_{\{Y-X^2\}}$ is closed and (MP) fails for $T_{\{Y-X^2\}}$.

Example 8.2 Let $S_1 = \{X, 1 - X, Y - 1, 1 - XY\}$, $S_2 = \{X, 1 - X, XY - 1\}$. By Corollary 5.5, M_{S_1} and M_{S_2} satisfy condition (‡).

Example 8.3 Let $S_1 = \{X, 1 - X, Y^2 - X^3\}$, $S_2 = \{X, 1 - X, Y^3 - X^2\}$. By Corollary 5.5, M_{S_1} satisfies (‡). By Theorem 5.1, T_{S_2} does not satisfy (SMP). By Corollary 6.4, M_{S_2} does satisfy (MP).

In [K–S] the authors ask for examples where (SMP) holds but (‡) does not hold, and for examples where $T^{\text{lin}}_S \neq T^{\ddagger}_S$. Although we are unable to settle this issue yet, we note that $T^{\text{lin}}_S = T^{\ddagger}_S$ does hold when $n = 1$. Also, in the special case where S is a set of linear polynomials, $T^{\text{lin}}_S = T^{\ddagger}_S$ holds when $n = 2$ (using Theorem 7.4 and [K–M, Th. 3.5]), and also when $n = 3$, except possibly in the case when K_S contains a 2-dimensional cone but does not contain a 3-dimensional cone. What happens in this exceptional case is not clear.

Also, in this regard, the following two examples are interesting.

Example 8.4 Let $S = \{X, Y, 1 - XY\}$. The polynomial XY is bounded on K_S and one checks that (SMP) holds for $T_{S \cup \{XY - \lambda, -(XY - \lambda)\}}$, for each $0 \leq \lambda \leq 1$. Thus by Theorem 1.5, (SMP) holds for T_S. It is not known if (‡) holds for T_S. This example is due to K. Schmüdgen.

We note the following:

Proposition 8.5 *Let S and $S' \subseteq \mathbb{R}[X]$ (any number of variables) and assume that $K_S = K_{S'}$. Assume that (SMP) holds for T_S and that $S \subseteq T_{S'}^{\text{lin}}$. Then (SMP) holds for $T_{S'}$.*

Proof: If $S \subseteq T_{S'}^{\text{lin}}$ then $T_S \subseteq T_{S'}^{\text{lin}}$ (since $T_{S'}^{\text{lin}}$ is a preordering), therefore $T_S^{\text{lin}} \subseteq T_{S'}^{\text{lin}}$. But $T_{S'}^{\text{alg}} = T_S^{\text{alg}} = T_S^{\text{lin}}$. It follows that $T_{S'}^{\text{alg}} = T_S^{\text{lin}}$. □

Example 8.6 In Example 8.2, the set K_{S_2} was defined using the set $S_2 = \{X, 1 - X, XY - 1\}$. But it can also be defined by the set $S_3 = \{1 - X, Y, XY - 1\}$. According to Theorem 1.5, (SMP) holds for T_{S_3}. Is it true that (‡) holds for T_{S_3}? This would be true by Theorem 5.3 if we could show that $X + N \in T_{S_3}$ for N sufficiently large. Unfortunately, we have the following:

Claim 1: If N is a positive integer, then $X + N \notin T_{S_3}$. Assuming by way of contradiction that $X + N \in T_{S_3}$, there is a representation

$$X + N = \sum_{i,j,k=0}^{1} \sigma_{ijk}(1 - X)^i (XY - 1)^j Y^k$$

with σ_{ijk} sums of squares in $\mathbb{R}[X, Y]$. Since the variable Y does not occur on the left hand side one concludes that $\sigma_{ijk} = 0$ if $j + k \geq 1$, hence

$$X + N = \sigma_0 + \sigma_1(1 - X),$$

where σ_0, σ_1 are sums of squares in $\mathbb{R}[X]$. But then $(1+\sigma_1)(X+N) = \sigma_0+(N+1)\sigma_1$, hence $X + N$ is a sum of squares in $\mathbb{R}(X)$ – which is false.

On the other hand, we have:

Claim 2: $X + \varepsilon Y \in T_{S_3}$ for every $0 < \varepsilon \in \mathbb{R}$. Note that $X + \varepsilon Y = (X + \frac{1}{n}Y) + (\varepsilon - \frac{1}{n})Y \in T_{S_3}$ if $1 \leq n \in \mathbb{N}$ such that $\frac{1}{n} \leq \varepsilon$ and $X + \frac{1}{n}Y \in T_{S_3}$. Thus it suffices to prove that there are arbitrarily large numbers $n \in \mathbb{N}$ with $nX + Y - 1 \in T_{S_3}$. The proof is by induction on n.

$n = 1$: $(XY - 1)(1 - X) = XY - 1 - X^2Y + X \in T_{S_3}$. Adding $X^2Y \in T_{S_3}$ and $Y(1 - X) \in T_{S_3}$ one gets $X + Y - 1 \in T_{S_3}$.

$n \Rightarrow n + 1$: $(n + 1)X + Y - 1 - nX^2 - XY = (nX + Y - 1)(1 - X) \in T_{S_3}$. Adding $nX^2 \in T_{S_3}$, $XY - 1 \in T_{S_3}$ and $1 \in T_{S_3}$ one obtains $(n+1)X + Y - 1 \in T_{S_3}$.

By Claim 2, $X \in T_{S_3}^{\ddagger}$, so $T_{S_2} \subseteq T_{S_3}^{\ddagger} \subseteq T_{S_3}^{\text{lin}}$. Using Proposition 8.5 this provides a second proof that (SMP) holds for T_{S_3}. The question of whether or not (‡) holds for T_{S_3} remains open.

9 Some Open Problems

1. In Theorem 5.3, is it possible to replace assumption (a) with the (apparently weaker) assumption that there exists N such that $N - \|X\|^2 \geq 0$ on K_S? Note: If the answer is 'no' then we have an example where (SMP) holds for T_S but (‡) fails for T_S. This would answer Open Problem 3 in [K–M, Sect. 6].

2. In Theorem 5.3, are the strong assumptions (b) on the fibers actually necessary? If they are necessary then again we have the answer to Open Problem 3 in [K–M].

3. Does the converse of Theorem 1.5 hold? This seems unlikely. By Theorem 5.1, it is true when the fibers are linear. To the extent that the converse of Theorem 1.5 is true, it provides a way of studying the question of when (SMP) and (MP) hold by induction on the dimension.

4. Suppose $n = 2$, $S = \{X, Y, 1-(X-1)(Y-1)\}$. Does (SMP) hold for T_S? Note: The only polynomials which are bounded on K_S are the constants, so Theorem 1.5 does not apply in this case. This problem is due to K. Schmüdgen.

5. Is it true that if $K_S = K_{S'}$, (‡) holds for T_S and $S \subseteq T_{S'}^{\ddagger}$ then (‡) holds for $T_{S'}$. (Compare to Proposition 8.5.) This seems unlikely.

6. For $n = 1$, does there exist a finite set S in $\mathbb{R}[X]$ with K_S compact and $M_S \neq T_S$? In particular, does there exist a finite set S in $\mathbb{R}[X]$ with K_S compact, T_S saturated, and $M_S \neq T_S$? These questions are of interest already in the case $|S| = 2$. Note: If K_S is compact, then $M_S^{\text{lin}} = T_S^{\text{alg}}$, so $M_S = T_S$ holds whenever M_S is closed.

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