Lower bounds for a polynomial in terms of its coefficients

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Abstract. We determine new sufficient conditions in terms of the coefficients for a polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ ($d \geq 1$) in $n \geq 1$ variables to be a sum of squares of polynomials, thereby strengthening results of Fidalgo and Kovacec [2] and of Lasserre [6]. Exploiting these results, we determine, for any polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ whose highest degree term is an interior point in the cone of sums of squares of forms of degree d, a real number r such that $f - r$ is a sum of squares of polynomials. The existence of such a number r was proved earlier by Marshall [8], but no estimates for r were given. We also determine a lower bound for any polynomial f whose highest degree term is positive definite.

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1. Introduction

Fix a non-constant polynomial $f \in \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \cdots, X_n]$, where $n \geq 1$ is an integer number, and define

$$f_* = \inf\{f(\underline{a}) \mid \underline{a} \in \mathbb{R}^n\}.$$

Let $\deg(f) = m$ and decompose f as $f = f_0 + \cdots + f_m$ (the homogeneous decomposition of f), where f_i is a form of degree i, $i = 0, \ldots, m$. A necessary condition for $f_* \neq -\infty$ is that f_m is positive semidefinite (so, in particular, m is even). A sufficient condition for $f_* \neq -\infty$ is that f_m is positive definite.

We assume from now on that $\deg(f) = 2d$, $d \geq 1$, i.e., $m = 2d$. For convenience we denote the cone of all sos[1] polynomials by $\sum \mathbb{R}[\underline{X}]^2$ and the cone of all positive semidefinite forms and sos forms of degree $2d$ by $P_{2d,n}$ and $\Sigma_{2d,n}$. Also,

[1] Abbreviation for sum of squares

by $\mathbb{R}[\underline{X}]_k$ we mean the subspace of $\mathbb{R}[\underline{X}]$ consisting of all polynomials with degree at most k.

Define

$$f_{sos} = \sup\{r \in \mathbb{R} \mid f - r \in \sum \mathbb{R}[\underline{X}]^2\}. \tag{1}$$

One can prove that $f_{sos} \leq f_*$ and

$$f_{sos} = \inf\{\ell(f) \mid \ell \in \chi_{2d}\}, \tag{2}$$

where χ_{2d} is the set of all linear maps $\ell : \mathbb{R}[\underline{X}]_{2d} \to \mathbb{R}$ such that $\ell(1) = 1$ and $\ell(p^2) \geq 0$ for all $p \in \mathbb{R}[\underline{X}]$ of degree $\leq d$. Computing f_* is difficult in general, and one of the successful approaches is to compute f_{sos} instead. This is accomplished by using *semidefinite programming* (SDP) which is a polynomial time algorithm [5] [9]. The equivalent definitions (1) and (2) for f_{sos} can be considered as two SDP problems which are dual of each other (and the duality gap is 0 in this case).

When is a given polynomial $f \in \mathbb{R}[\underline{X}]$ a sum of squares? One obvious necessary condition is that $f \geq 0$ on $\mathbb{R}^n$, but there is a well known result due to Hilbert [3] that, in general, for $n \geq 3$, $d \geq 2$ and $(n, d) \neq (3, 2)$, $P_{2d,n} \neq \Sigma_{2d,n}$. The Motzkin polynomial $s(X, Y) = 1 - 3X^2Y^2 + X^4Y^2 + X^2Y^4$ is a concrete example of a polynomial which is non-negative on $\mathbb{R}^2$ but is not a sum of squares.

In this paper we are interested in some recent results, due to Lasserre [6] and to Fidalgo and Kovacec [2], which give sufficient conditions on the coefficients for a polynomial to be a sum of squares. We establish new and improved versions of these results; see Ths. 2.1 and 2.3 and Cors. 2.2 and 2.4.

We denote by $P^{\circ}_{2d,n}$ and $\Sigma^{\circ}_{2d,n}$ the interior of $P_{2d,n}$ and $\Sigma_{2d,n}$, more precisely, the interior in the subspace of $\mathbb{R}[\underline{X}]$ consisting of forms of degree $2d$. A necessary condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma_{2d,n}$. A sufficient condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma^{\circ}_{2d,n}$ [8, Prop. 5.1].

Scaling $\underline{X}$ by a non-zero scalar k does not change the value of f_{sos}. In Section 3 we show that if $f_{2d} = \sum_{i=1}^{n} X_i^{2d}$ there are suitable choices for k and r such that $f(k\underline{X}) - r$ satisfies the sufficient conditions established in Cors. 2.2 and 2.4. This allows us to determine, assuming that $f_{2d} \in \Sigma^{\circ}_{2d,n}$, two lower bounds for f_{sos}, which we denote by r_L and r_{FK} respectively; see Ths. 3.1 and 3.2. Yet another lower bound for f_{sos}, which we denote by r_{dmt}, is obtained by applying [2, Th. 2.3] more or less directly; see Th. 3.3. The bounds r_L, r_{FK} and r_{dmt} are not comparable; see Ex. 4.2. If we assume only that $f_{2d} \in P^{\circ}_{2d,n}$ then it is still possible to determine lower bounds r_L, r_{FK} and r_{dmt} for f_*, but these may not be lower bounds for f_{sos}; see Th. 4.3.

We introduce more notation that we will need. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. For $\underline{X} = (X_1, \ldots, X_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $\underline{X}^{\alpha} := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Clearly, using this notation, every polynomial $f \in \mathbb{R}[\underline{X}]$ can be written as $f(\underline{X}) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} \underline{X}^{\alpha}$, where $f_{\alpha} \in \mathbb{R}$ and $f_{\alpha} = 0$, except for finitely many α. Let $\Omega(f) = \{\alpha \in \mathbb{N}^n \mid f_{\alpha} \neq$

$0\} \setminus \{\underline{0}, 2d\epsilon_1, \ldots, 2d\epsilon_n\}$, where $2d = \deg(f)$, $\epsilon_i = (\delta_{i1}, \ldots, \delta_{in})$, and

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

We denote $f_{\underline{0}}$ and $f_{2d\epsilon_i}$ by f_0 and $f_{2d,i}$ for short. Thus f has the form

$$f = f_0 + \sum_{\alpha \in \Omega(f)} f_\alpha \underline{X}^\alpha + \sum_{i=1}^{n} f_{2d,i} X_i^{2d}. \tag{3}$$

Let $\Delta(f) = \{\alpha \in \Omega(f) \mid f_\alpha \underline{X}^\alpha$ is not a square in $\mathbb{R}[\underline{X}]\} = \{\alpha \in \Omega(f) \mid$ either $f_\alpha < 0$ or α_i is odd for some $i \in \{1, \ldots, n\}\}$. Since our polynomial f is usually fixed, we will often denote $\Omega(f)$ and $\Delta(f)$ just by Ω and Δ for short.

Let $\widetilde{f}(\underline{X}, Y) = Y^{2d} f(\frac{X_1}{Y}, \ldots, \frac{X_n}{Y})$. From (3), it is clear that

$$\widetilde{f}(\underline{X}, Y) = f_0 Y^{2d} + \sum_{\alpha \in \Omega} f_\alpha \underline{X}^\alpha Y^{2d-|\alpha|} + \sum_{i=1}^{n} f_{2d,i} X_i^{2d}$$

is a form of degree $2d$, called the homogenization of f. We have the following well-known result:

Proposition 1.1. *f is sos if and only if $\widetilde{f}$ is sos.*

Proof. See [7, Prop. 1.2.4]. $\square$

For a (univariate) polynomial of the form $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero, we denote by $C(p)$ the unique positive root of p [10, Th. 1.1.3]. For any polynomial $q(t) = \sum_{i=0}^{n} b_i t^i$, $b_n \neq 0$, the roots of q are bounded in absolute value by $C(t^n - \sum_{i=0}^{n-1} \frac{|b_i|}{|b_n|} t^i)$. By convention, $C(t^n) := 0$.

There are various upper bounds for $C(p)$ which are expressible in an elementary way in terms of the coefficients of p, for example,

Proposition 1.2. *Suppose $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero. Then*

1. $C(p) \leq \max\{1, a_0 + a_1 + \cdots + a_{n-1}\}$,
2. $C(p) \leq \max\{a_0, 1 + a_1, 1 + a_2, \ldots, 1 + a_{n-1}\}$,
3. $C(p) \leq 2\max\{a_{n-1}, (a_{n-2})^{1/2}, (a_{n-3})^{1/3}, \ldots, (a_0)^{1/n}\}$.

Proof. Bounds (1) and (2) are due basically to Cauchy. See [1] for these bounds and for other bounds of this sort. See [4, Ex. 4.6.2: 20] for bound (3). $\square$

2. Sufficient conditions for a polynomial to be sos

In [2, Th. 2.3], Fidaldo and Kovacec prove that, for a form

$$E(\underline{X}) = \sum_{i=1}^{n} \beta_i X_i^{2d} - \mu \underline{X}^{\alpha}$$

such that $\alpha_i > 0$ and $\beta_i \geq 0$ for every $i = 1, \dots, n$ and $\mu \geq 0$ if all α_i are even, the following are equivalent:

i. E is positive semidefinite.

ii. $|\mu| \leq 2d \prod_{i=1}^{n} \left(\frac{\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{2d}}$.

iii. E is a sum of binomial squares, i.e., squares of the form $(a\underline{X}^{\sigma} - b\underline{X}^{\tau})^2$.

iv. E is sos.

We use this result to improve on the sufficient conditions given in [6, Th. 3] and [2, Th. 4.3]. We show that [6, Th. 3] can be improved as follows:

Theorem 2.1. *Suppose $f \in \mathbb{R}[\underline{X}]$ is a form of degree $2d$ and*

$$f_{2d,i} \geq \sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}, \quad i = 1, \dots, n.$$

Then f is a sum of (binomial) squares.

Proof. We claim that

$$\sum_{i=1}^{n} |f_\alpha| \frac{\alpha_i}{2d} X_i^{2d} + f_\alpha \underline{X}^{\alpha}$$

is sobs[2], for each $\alpha \in \Delta$. It suffices to show that $\sum\limits_{\alpha_i \neq 0} |f_\alpha| \frac{\alpha_i}{2d} X_i^{2d} + f_\alpha \underline{X}^{\alpha}$ is sobs, for each $\alpha \in \Delta$. Since

$$2d \prod_{\alpha_i \neq 0} \left(\frac{|f_\alpha| \frac{\alpha_i}{2d}}{\alpha_i}\right)^{\frac{\alpha_i}{2d}} = 2d \frac{|f_\alpha|}{2d} = |f_\alpha| \geq |f_\alpha|,$$

and since $f_\alpha < 0$ if all the α_i are even, by definition of Δ, this follows, as a consequence of [2, Th. 2.3]. This proves the claim. Adding, as α runs through Δ, this implies

$$\sum_{i=1}^{n} \Big(\sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}\Big) X_i^{2d} + \sum_{\alpha \in \Delta} f_\alpha \underline{X}^{\alpha}$$

is sobs. Since $f_{2d,i} \geq \sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}$, for each i,

$$\sum_{i=1}^{n} f_{2d,i} X_i^{2d} - \sum_{i=1}^{n} \Big(\sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}\Big) X_i^{2d} = \sum_{i=1}^{n} \Big(f_{2d,i} - \sum_{\alpha \in \Delta} |f_\alpha| \frac{\alpha_i}{2d}\Big) X_i^{2d}$$

[2]Abbreviation for sum of binomial squares

is sobs. Adding again, this implies that

$$\sum_{i=1}^{n} f_{2d,i} X_i^{2d} + \sum_{\alpha\in\Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Finally, since the remaining terms $f_\alpha \underline{X}^\alpha$, $\alpha \in \Omega\backslash\Delta$, are squares of monomials, by definition of Δ, this implies that f is sobs. $\square$

Corollary 2.2. *For any polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$, if*

$$\text{(L1)} \quad f_0 \geq \sum_{\alpha\in\Delta} |f_\alpha| \frac{2d-|\alpha|}{2d} \quad \textit{and} \quad \text{(L2)} \quad f_{2d,i} \geq \sum_{\alpha\in\Delta} |f_\alpha| \frac{\alpha_i}{2d}, \quad i = 1,\dots,n,$$

then f is a sum of squares.

Proof. Apply Th. 2.1 to the homogenization $\tilde{f}$ of f to conclude that $\tilde{f}$ is sos. Consequently, by Prop. 1.1, f is also sos. $\square$

In [6, Th. 3], it is proved that if

$$f_0 \geq \sum_{\alpha\in\Delta} |f_\alpha| \text{ and } f_{2d,i} \geq \sum_{\alpha\in\Delta} |f_\alpha| \frac{|\alpha|}{2d}, \quad i = 1,\dots,n,$$

then f is a sum of squares. Since $1 \geq \frac{2d-|\alpha|}{2d}$ and $\frac{|\alpha|}{2d} \geq \frac{\alpha_i}{2d}$, it is clear that Cor. 2.2 improves on [6, Th. 3].

We explain now how [2, Th. 4.3] can be improved.

Theorem 2.3. *Suppose $f \in \mathbb{R}[\underline{X}]$ is a form of degree $2d$ and*

$$\min_{i=1,\dots,n} f_{2d,i} \geq \frac{1}{2d} \sum_{\alpha\in\Delta} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}.$$

Then f is a sum of (binomial) squares.

Here, $\alpha^\alpha := \alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}$ (the convention being that $0^0 := 1$).

Proof. Let $e_\alpha := \frac{1}{2d} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}$. We claim that

$$e_\alpha \sum_{i=1}^{n} X_i^{2d} + f_\alpha \underline{X}^\alpha$$

is sobs, for each $\alpha \in \Delta$. Since $e_\alpha \geq 0$, $e_\alpha \sum_{\alpha_i=0} X_i^{2d}$ is sobs, so it suffices to show that $e_\alpha \sum_{\alpha_i\neq 0} X_i^{2d} - f_\alpha \underline{X}^\alpha$ is sobs. Since

$$2d \prod_{\alpha_i\neq 0} \left(\frac{e_\alpha}{\alpha_i}\right)^{\frac{\alpha_i}{2d}} = \frac{2d e_\alpha}{(\alpha^\alpha)^{\frac{1}{2d}}} = |f_\alpha| \geq |f_\alpha|,$$

and since $f_\alpha < 0$ if all the α_i are even, by definition of Δ, this follows from [2, Th. 2.3]. This proves the claim. Adding, this implies

$$\sum_{\alpha\in\Delta} e_\alpha \sum_{i=1}^{n} X_i^{2d} + \sum_{\alpha\in\Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Since $f_{2d,i} \ge \sum_{\alpha\in\Delta} e_\alpha$, for each i,

$$\sum_{i=1}^n f_{2d,i}X_i^{2d} - \sum_{\alpha\in\Delta} e_\alpha \sum_{i=1}^n X_i^{2d} = \sum_{i=1}^n (f_{2d,i} - \sum_{\alpha\in\Delta} e_\alpha)X_i^{2d}$$

is sobs. Adding again, this implies

$$\sum_{i=1}^n f_{2d,i}X_i^{2d} + \sum_{\alpha\in\Delta} f_\alpha \underline{X}^\alpha$$

is sobs. Finally, since the remaining terms $f_\alpha\underline{X}^\alpha$, $\alpha \in \Omega\backslash\Delta$, are squares of monomials, this implies f is sobs. □

In [2, Th. 4.3] it is proved that if $f \in \mathbb{R}[\underline{X}]$ is any form of degree $2d$ and

$$\min_{i=1,\dots,n} f_{2d,i} \ge \frac{1}{n}(\frac{n}{2d})^{2d} \sum_{\alpha\in\Delta} |f_\alpha|\alpha^\alpha$$

then f is a sum of squares. Using $\alpha^\alpha \ge (\frac{2d}{n})^{2d}$, one sees immediately that

$$\frac{1}{n}(\frac{n}{2d})^{2d}\alpha^\alpha \ge \frac{1}{2d}(\alpha^\alpha)^{\frac{1}{2d}}.$$

Consequently, Th. 2.3 improves on [2, Th. 4.3]. The fact that $\alpha^\alpha \ge (\frac{2d}{n})^{2d}$ is an immediate consequence of the fact that the minimum value of the function

$$G(t_1,\dots,t_n) := t_1^{t_1}\cdots t_n^{t_n}$$

on the compact subset of $\mathbb{R}^n$ defined by $t_i \ge 0$, $i = 1,\dots,n$, $\sum_{i=1}^n t_i = 2d$ is equal to $(\frac{2d}{n})^{2d}$, the minimum occurring at the point $t_1 = \cdots = t_n = \frac{2d}{n}$.

Corollary 2.4. *If $f \in \mathbb{R}[\underline{X}]$ is a polynomial of degree $2d$ and*

$$\text{(FK)} \qquad \min_{i=1,\dots,n}\{f_{2d,i}, f_0\} \ge \tfrac{1}{2d}\sum_{\alpha\in\Delta} |f_\alpha|(\alpha^\alpha)^{\frac{1}{2d}}(2d-|\alpha|)^{\frac{2d-|\alpha|}{2d}}$$

then f is a sum of squares.

Proof. Homogenize f and apply Th. 2.3 and Prop. 1.1. □

Recall that $\Sigma^\circ_{2d,n}$ (resp., $P^\circ_{2d,n}$) denotes the interior of the cone $\Sigma_{2d,n}$ (resp., $P_{2d,n}$) in the real vector space consisting of forms of degree $2d$. The following result is well-known. It is proved, for example, in [8, Prop. 5.3(2)].

Corollary 2.5. $X_1^{2d} + \cdots + X_n^{2d} \in \Sigma^\circ_{2d,n}$.

Proof. Let $f(\underline{X}) = X_1^{2d} + \cdots + X_n^{2d} + h(\underline{X})$ where $h(\underline{X})$ is any form of degree $2d$ whose coefficients have absolute value $\le \epsilon$ where ϵ is some small positive real. Applying Th. 2.1 or Th. 2.3, one sees that f is sos, for ϵ sufficiently small. □

Remark 2.6. Let C be a cone in a finite dimensional real vector space V. Let C° denote the interior of C. If $f \in C^\circ$ and $g \in V$ then $g \in C^\circ$ iff $g - \epsilon f \in C$ for some real $\epsilon > 0$.

Proof. Suppose $g - \epsilon f \in C$. Let $h \in V$. Since f belongs to the interior of C, there exists some real $\delta > 0$ such that $f + \frac{\delta}{\epsilon} h \in C$. Then $g + \delta h = (g - \epsilon f) + \epsilon(f + \frac{\delta}{\epsilon} h) \in C$. This proves that g belongs to the interior of C. The other implication is clear. □

It follows from Cor. 2.5 and Rem. 2.6 that a form f of degree $2d$ is an interior point of $\Sigma_{2d,n}$ iff $f - \epsilon \sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$ for some real $\epsilon > 0$. Ths. 2.1 and 2.3 provide sufficient conditions for $f \in \Sigma_{2d,n}^\circ$ to hold and have the nice additional property of allowing computation of ϵ:

Corollary 2.7. *If f is a form of degree $2d$ and $\epsilon := \max\{\epsilon_1, \epsilon_2\} > 0$ where*

$$\epsilon_1 := \min_{i=1,\dots,n} (f_{2d,i} - \sum_{\alpha\in\Delta} |f_\alpha| \frac{\alpha_i}{2d}),\ \ \epsilon_2 := \min_{i=1,\dots,n} f_{2d,i} - \frac{1}{2d} \sum_{\alpha\in\Delta} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}},$$

then $f \in \Sigma_{2d,n}^\circ$ and $f - \epsilon \sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$.

Proof. Applying Th. 2.1 or Th. 2.3 (depending on whether $\epsilon = \epsilon_1$ or $\epsilon = \epsilon_2$) to the form $f - \epsilon \sum_{i=1}^n X_i^{2d}$, we see that $f - \epsilon \sum_{i=1}^n X_i^{2d}$ is sos. □

3. Determining lower bounds

Every polynomial f of degree $2d$ decomposes as $f = f_0 + f_1 + \cdots + f_{2d}$ (the homogeneous decomposition of f) where each f_i is a form of degree i. The following theorem gives a sufficient condition for $f_{sos} \neq -\infty$ in terms of the highest degree form, f_{2d}, and a concrete lower bound for f_{sos}:

Theorem 3.1. *If $f_{2d} \in \Sigma_{2d,n}^\circ$ then $f_{sos} \geq r_L$, where*

$$r_L := f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|},$$
$$k := \max_{i=1,\dots,n} C(t^{2d} - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{\alpha_i}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|})$$

and $\epsilon > 0$ is such that $f_{2d} - \epsilon(\sum_{i=1}^n X_i^{2d}) \in \Sigma_{2d,n}$.

Note: (i) If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $k = C(t^{2d}) := 0$ and $r_L := f_0$. (ii) Th. 3.1 proves in particular that if $f_{2d} \in \Sigma_{2d,n}^\circ$ then $f_{sos} \neq -\infty$, i.e., it provides another proof of [8, Prop. 5.1]. (iii) If $\ell \geq k$ then

$$f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|} \leq f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|} = r_L.$$

In this way, by taking ℓ to be an upper bound for k computed using Prop. 1.2, we obtain a lower bound $f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d-|\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|}$ for f_{sos} which is expressible in an elementary way in terms of ϵ and the coefficients f_α, $|\alpha| < 2d$.

Proof. Since $f_{2d} \in \Sigma^{\circ}_{2d,n}$, by Cor. 2.5 and Rem. 2.6, there exists $\epsilon > 0$ such that $f_{2d} = \epsilon(X_1^{2d} + \cdots + X_n^{2d}) + g$ for some $g \in \Sigma_{2d,n}$. Scaling suitably ($X_i \mapsto \frac{X_i}{\sqrt[2d]{\epsilon}}$), we can assume that $\epsilon = 1$. Let $\hat{f} := f - g$. Decomposing $\hat{f}$ as in equation (3) yields

$$\hat{f} = f_0 + \sum_{\alpha \in \Omega, |\alpha| < 2d} f_\alpha \underline{X}^\alpha + \sum_{i=1}^{n} X_i^{2d}. \tag{4}$$

If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $\hat{f} - r_L = \hat{f} - f_0$ is sos, using equation (4) and the definition of Δ, so $f - r_L$ is also sos and the result is clear. Thus we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Scaling by $X_i \mapsto kX_i$, and rewriting condition (L2) of Cor. 2.2 for the polynomial $\hat{f}(k\underline{X}) - r$, using equation (4), yields

$$k^{2d} \geq \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{\alpha_i}{2d} k^{|\alpha|}, \quad i = 1, \ldots, n.$$

By definition of k, $k^{2d} \geq \sum_{\alpha \in \Delta, |\alpha| < 2d} |f_\alpha| \frac{\alpha_i}{2d} k^{|\alpha|}$ for all i, so condition (L2) holds for $\hat{f}(k\underline{X}) - r$. Rewriting condition (L1) of Cor. 2.2 for the polynomial $\hat{f}(k\underline{X}) - r$, we see that if $r \leq r_L$ then (L1) holds for $\hat{f}(k\underline{X}) - r$ so $\hat{f} - r$ is sos and hence also $f - r$ is sos. □

Applying Cor. 2.4 to the polynomial $f - r$ we see that, for any real r, $f - r$ is sos as long as

$$\min_{i=1,\cdots,n} \{f_{2d,i}, f_0 - r\} \geq \text{ the right side of (FK).}$$

One can use the same idea used in the proof of Th. 3.1 to find a lower bound for f_{sos} using Cor. 2.4 as follows:

Theorem 3.2. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then $f_{sos} \geq r_{FK}$, where $r_{FK} := f_0 - k^{2d}$, $k := C(t^{2d} - \sum_{i=1}^{2d-1} b_i t^i)$,*

$$b_i := \frac{1}{2d}(2d-i)^{\frac{2d-i}{2d}} \epsilon^{-\frac{i}{2d}} \sum_{\alpha \in \Delta, |\alpha| = i} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}, \ i = 1, \ldots, 2d-1$$

and $\epsilon > 0$ is given as in Th. 3.1.

Note: If $\ell \geq k$ then

$$f_0 - \sum_{i=1}^{2d-1} b_i \ell^i \leq f_0 - \sum_{i=1}^{2d-1} b_i k^i = f_0 - k^{2d} = r_{FK}$$

so, using Prop. 1.2 again, we get another lower bound for f_{sos} expressible in an elementary way in terms of ϵ and the coefficients f_α, $|\alpha| < 2d$.

Proof. After scaling we can assume that $\epsilon = 1$ and $f_{2d} = X_1^{2d} + \cdots + X_n^{2d} + g$, where $g \in \Sigma_{2d,n}$. If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $b_i = 0$ for $i = 1, \ldots 2d-1$, $k = 0$ (by definition of $C(t^{2d})$), so $r_{FK} = f_0$. In this case the result is clear. So we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Set $r = r_{FK}$. Rewriting condition (FK) for the polynomial $\hat{f}(k\underline{X}) - r$, where $\hat{f} := f - g$, yields the condition:

$$\min\{(f_0 - r), k^{2d}\} \geq \sum_{i=1}^{2d-1} b_i k^i. \tag{5}$$

By definition of k and r, (5) holds, in fact, $f_0 - r = k^{2d} = \sum_{i=1}^{2d-1} b_i k^i$. This proves that $\hat{f} - r$ is sos and hence also that $f - r$ is sos. □

One can also apply [2, Th. 2.3] directly to obtain a lower bound for f_{sos}.

Theorem 3.3. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then*

$$f_{sos} \geq r_{dmt} := f_0 - \sum_{\alpha \in \Delta, |\alpha| < 2d} (2d - |\alpha|) \left[\left(\frac{f_\alpha}{2d} \right)^{2d} ((\frac{t}{\epsilon})^{|\alpha|} \alpha^\alpha) \right]^{\frac{1}{2d-|\alpha|}},$$

where $t := |\{\alpha \in \Delta \mid |\alpha| < 2d\}|$ and $\epsilon > 0$ is given as in Th. 3.1.

Proof. Let $\Delta' = \{\alpha \in \Delta \mid |\alpha| < 2d\}$. After scaling, we can assume that $\epsilon = 1$. Let $\overline{f} = f_0 + \sum_{\alpha \in \Delta'} f_\alpha \underline{X}^\alpha + X_1^{2d} + \cdots + X_n^{2d}$ and let $F(\underline{X}, Y)$ denote the homogenization of $\overline{f}(\sqrt[2d]{t}\underline{X}) - r$, where $r := f_0 - \sum_{\alpha \in \Delta'} r_\alpha$, each $r_\alpha \geq 0$. Then

$$\begin{aligned} F(\underline{X}, Y) &= (f_0 - r)Y^{2d} + \sum_{\alpha \in \Delta'} (X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d} \underline{X}^\alpha Y^{2d-|\alpha|}) \\ &= \sum_{\alpha \in \Delta'} (r_\alpha Y^{2d} + X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d} \underline{X}^\alpha Y^{2d-|\alpha|}). \end{aligned}$$

By [2, Th. 2.3], each term appearing in this sum will be sos if

$$|f_\alpha| t^{\frac{|\alpha|}{2d}} \leq 2d \left(\frac{r_\alpha}{2d - |\alpha|} \right)^{\frac{2d-|\alpha|}{2d}} \prod_{\alpha_i \neq 0} \left(\frac{1}{\alpha_i} \right)^{\frac{\alpha_i}{2d}},$$

or, equivalently, if

$$r_\alpha \geq (2d - |\alpha|) \left[\left(\frac{f_\alpha}{2d} \right)^{2d} t^{|\alpha|} \alpha^\alpha \right]^{\frac{1}{2d-|\alpha|}}.$$

Hence if $r \leq r_{dmt}$ then $\overline{f} - r$ is sos, so also $f - r$ is sos. □

4. Further remarks

(1) The sufficient conditions given in Ths. 2.1 and 2.3 are not comparable. These conditions are also not necessary.

Example 4.1.

(a) $f(X,Y,Z) = X^4 + Y^4 + 4Z^4 + 4XZ^3$ is sos, by Th. 2.1, but Th. 2.3 does not apply.

(b) $f(X,Y,Z) = X^4 + Y^4 + Z^4 + \sqrt{8}XYZ^2$ is sos, by Th. 2.3, but Th. 2.1 does not apply.

(c) $f(X,Y,Z) = 16X^4 + Y^4 + 4Z^4 + 8XZ^3$ is sos, but neither Th. 2.1 nor Th. 2.3 applies.

(2) The bounds r_L, r_{FK} and r_{dmt} described in Ths. 3.1, 3.2 and 3.3 are not comparable.

Example 4.2.

(a) If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $r_L = r_{FK} = r_{dmt} = f_0$.

(b) For $f(X,Y) = X^6 + Y^6 + 7XY - 2X^2 + 7$, we have $r_L \approx -1.124$, $r_{FK} \approx -0.99$ and $r_{dmt} \approx -1.67$, so $r_{FK} > r_L > r_{dmt}$.

(c) For $f(X,Y) = X^6 + Y^6 + 4XY + 10Y + 13$, $r_L \approx -0.81$, $r_{FK} \approx -0.93$ and $r_{dmt} \approx -0.69$, so $r_{dmt} > r_L > r_{FK}$.

(d) For $f(X,Y) = X^4 + Y^4 + XY - X^2 - Y^2 + 1$, $r_L \approx -0.125$, $r_{FK} \approx -0.832$ and $r_{dmt} \approx -0.875$, so $r_L > r_{FK} > r_{dmt}$.

(3) To be able to compute r_L, r_{FK} and r_{dmt} one needs to know ϵ and the coefficients f_α, $|\alpha| < 2d$. What can one do if ϵ is not given, i.e., if only the coefficients f_α, $|\alpha| \le 2d$ are given? Applying Cor. 2.7 to the form f_{2d} allows us to compute ϵ in certain cases: If $\epsilon := \max\{\epsilon_1, \epsilon_2\} > 0$ where

$$\begin{aligned} \epsilon_1 &:= \min_{i=1,\dots,n} (f_{2d,i} - \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha| \tfrac{\alpha_i}{2d}), \\ \epsilon_2 &:= \min_{i=1,\dots,n} f_{2d,i} - \tfrac{1}{2d} \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha| (\alpha^\alpha)^{\frac{1}{2d}}, \end{aligned}$$

then $f_{2d} \in \Sigma^\circ_{2d,n}$ and $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$.

(4) So far we have been assuming that $f_{2d} \in \Sigma^\circ_{2d,n}$ and we have used this assumption to determine lower bounds for f_{sos}. What can one say if one assumes only that $f_{2d} \in P^\circ_{2d,n}$? Suppose $\epsilon > 0$ is given such that $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in P_{2d,n}$. One can then define r_L exactly as in Th. 3.1, but using this new ϵ, i.e.,

$$\begin{aligned} r_L &:= f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{2d - |\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|}, \\ k &:= \max_{i=1,\dots,n} C(t^{2d} - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{\alpha_i}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|}). \end{aligned}$$

The r_L defined in this way might not be a lower bound for f_{sos} (it is even possible that $f_{sos} = -\infty$), but it will be a lower bound for f_*. Similar remarks apply to the other bounds r_{FK} and r_{dmt}.

Theorem 4.3. *If $f_{2d} \in P^{\circ}_{2d,n}$ and $\epsilon > 0$ is such that $f_{2d} - \epsilon \sum_{i=1}^{n} X_i^{2d} \in P_{2d,n}$ then r_L, r_{FK} and r_{dmt}, defined as in Ths.* 3.1, 3.2 *and* 3.3*, respectively, but using this new choice of ϵ, are lower bounds for f on $\mathbb{R}^n$.*

Proof. Argue as in the proof of Ths. 3.1, 3.2 and 3.3. The form g is no longer sos but it is positive semidefinite, which is all one needs for the conclusion. □

Note: In Th. 4.3, the largest possible choice for ϵ is the minimum value of the rational function $f_{2d}/\sum_{i=1}^{n} X_i^{2d}$ on the $n-1$-sphere

$$\mathbb{S}^{n-1} := \{\underline{a} \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 = 1\}.$$

(5) We know that for any $p \in P^{\circ}_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_* \neq -\infty$ and, for any $p \in \Sigma^{\circ}_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_{sos} \neq -\infty$. Note that if $p \in P_{2d,n}$ is not positive definite then there exists $\underline{0} \neq \underline{a} \in \mathbb{R}^n$ such that $p(\underline{a}) = 0$. Let $g(\underline{X}) = \sum_{i=1}^{n} a_i X_i$. Then $(p+g)(t\underline{a}) = t\|\underline{a}\|^2 \to -\infty$ as $t \to -\infty$, so $(p+g)_* = -\infty$. Therefore for any $p \in \partial P_{2d,n}$ ($\partial P_{2d,n}$ denotes the boundary of $P_{2d,n}$, i.e. $\partial P_{2d,n} = P_{2d,n} \setminus P^{\circ}_{2d,n}$), there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$, such that $(p+g)_* = -\infty$. The validity of the corresponding result for boundary points of $\Sigma_{2d,n}$ is unknown to the authors.

Question 4.4. Is it true that for any $p \in \partial\Sigma_{2d,n}$ there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$ such that $(p+g)_{sos} = -\infty$?

The answer to this question is 'yes' if $n \leq 2$ or $d = 1$ or ($n = 3$ and $d = 2$) by Hilbert's result [3]. In fact these are precisely the cases where $P_{2d,n}$ and $\Sigma_{2d,n}$ coincide.

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