LOWER BOUNDS FOR A POLYNOMIAL IN TERMS OF ITS COEFFICIENTS

MEHDI GHASEMI AND MURRAY MARSHALL

Abstract. Recently Lasserre [6] gave a sufficient condition in terms of the coefficients for a polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ ($d \geq 1$) in $n \geq 1$ variables to be a sum of squares of polynomials. Exploiting this result, we are able to determine, for any polynomial $f \in \mathbb{R}[\underline{X}]$ of degree $2d$ whose highest degree term is an interior point in the cone of sums of squares of forms of degree d, a real number r such that $f - r$ is a sum of squares of polynomials. The existence of such a number r was proved earlier by Marshall [8], but no estimates for r were given. We also determine a lower bound for any polynomial f whose highest degree term is positive definite. Similar arguments are applied to results of Fidalgo and Kovacec [2], to determine other numbers r such that $f - r$ is a sum of squares and other lower bounds for f.

1. Introduction

Fix a non-constant polynomial $f \in \mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \cdots, X_n]$, where $n \geq 1$ is an integer number, and define

$$f_* = \inf\{f(\underline{a}) \mid \underline{a} \in \mathbb{R}^n\}.$$

Let $\deg(f) = m$ and decompose f as $f = f_0 + \cdots + f_m$ (the homogeneous decomposition of f), where f_i is a form of degree i, $i = 0, \ldots, m$. A necessary condition for $f_* \neq -\infty$ is that f_m is positive semidefinite (so, in particular, m is even). A sufficient condition for $f_* \neq -\infty$ is that f_m is positive definite.

We assume from now on that $\deg(f) = 2d$, $d \geq 1$, i.e., $m = 2d$. For convenience we denote the cone of all sos[1] polynomials by $\sum \mathbb{R}[\underline{X}]^2$ and the cone of all positive semidefinite forms and sos forms of degree $2d$ by $P_{2d,n}$ and $\Sigma_{2d,n}$. Also, by $\mathbb{R}[\underline{X}]_k$ we mean the subspace of $\mathbb{R}[\underline{X}]$ consisting of all polynomials with degree at most k.

Define

$$f_{sos} = \sup\{r \in \mathbb{R} \mid f - r \in \sum \mathbb{R}[\underline{X}]^2\}. \tag{1}$$

One can prove that $f_{sos} \leq f_*$ and

$$f_{sos} = \inf\{\ell(f) \mid \ell \in \chi_{2d}\}, \tag{2}$$

where χ_{2d} is the set of all linear maps $\ell : \mathbb{R}[\underline{X}]_{2d} \to \mathbb{R}$ such that $\ell(1) = 1$ and $\ell(p^2) \geq 0$ for all $p \in \mathbb{R}[\underline{X}]$ of degree $\leq d$. Computing f_* is difficult in general, and one of the successful approaches is to compute f_{sos} instead. This is accomplished by using *semidefinite programming* (SDP) which is a polynomial time algorithm [5] [9]. The equivalent definitions (1) and (2) for f_{sos} can be considered as two SDP problems which are dual of each other (and the duality gap is 0 in this case).

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[1]Abbreviation for sum of squares

When is a given polynomial $f \in \mathbb{R}[\underline{X}]$ a sum of squares? One obvious necessary condition is that $f \geq 0$ on $\mathbb{R}^n$, but there is a well known result due to Hilbert [3] that, in general, for $n \geq 3$, $d \geq 2$ and $(n, d) \neq (3, 2)$, $P_{2d,n} \neq \Sigma_{2d,n}$. The Motzkin polynomial $s(X, Y) = 1 - 3X^2Y^2 + X^4Y^2 + X^2Y^4$ is a concrete example of a polynomial which is non-negative on $\mathbb{R}^2$ but is not a sum of squares. In this paper we will be interested in some recent results, due to Lasserre [6] and to Fidalgo and Kovacec [2], which give sufficient conditions on the coefficients for a polynomial to be a sum of squares.

We denote by $P^\circ_{2d,n}$ and $\Sigma^\circ_{2d,n}$ the interior of $P_{2d,n}$ and $\Sigma_{2d,n}$, more precisely, the interior in the subspace of $\mathbb{R}[\underline{X}]$ consisting of forms of degree $2d$. A necessary condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma_{2d,n}$. A sufficient condition for $f_{sos} \neq -\infty$ is that $f_{2d} \in \Sigma^\circ_{2d,n}$ [8, Prop. 5.1].

Scaling $\underline{X}$ by a non-zero scalar k does not change the value of f_{sos}. In Sections 2 and 3 we show that if $f_{2d} = \sum_{i=1}^n X_i^{2d}$ there are suitable choices for k and r such that $f(k\underline{X}) - r$ satisfies Lasserre's condition in [6, Th. 3] and (after homogenization) Fidalgo-Kovacec's condition in [2, Th. 4.3]. This allows us to determine, assuming that $f_{2d} \in \Sigma^\circ_{2d,n}$, two lower bounds for f_{sos}, which we denote by r_L and r_{FK} respectively; see Ths. 2.3 and 3.1. Yet another lower bound for f_{sos}, which we denote by r_{dmt}, is obtained by applying [2, Th. 2.3] more or less directly; see Th. 3.2. The bounds r_L, r_{FK} and r_{dmt} are not comparable; see Ex. 4.1. If we assume only that $f_{2d} \in P^\circ_{2d,n}$ then it is still possible to determine lower bounds r_L, r_{FK} and r_{dmt} for f_*, but these may not be lower bounds f_{sos}; see Th. 4.3.

We introduce more notation that we will need. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers. For $\underline{X} = (X_1, \ldots, X_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $\underline{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Clearly, using this notation, every polynomial $f \in \mathbb{R}[\underline{X}]$ can be written as $f(\underline{X}) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \underline{X}^\alpha$, where $f_\alpha \in \mathbb{R}$ and $f_\alpha = 0$, except for finitely many α. Let $\Omega(f) = \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\} \setminus \{\underline{0}, 2d\epsilon_1, \ldots, 2d\epsilon_n\}$, where $2d = \deg(f)$, $\epsilon_i = (\delta_{i1}, \ldots, \delta_{in})$, and

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

We denote $f_{\underline{0}}$ and $f_{2d\epsilon_i}$ by f_0 and $f_{2d,i}$ for short. Thus f has the form

$$f = f_0 + \sum_{\alpha \in \Omega(f)} f_\alpha \underline{X}^\alpha + \sum_{i=1}^n f_{2d,i} X_i^{2d}. \tag{3}$$

Let $\Delta(f) = \{\alpha \in \Omega(f) \mid f_\alpha \underline{X}^\alpha$ is not a square in $\mathbb{R}[\underline{X}]\} = \{\alpha \in \Omega(f) \mid$ either $f_\alpha < 0$ or α_i is odd for some $i \in \{1, \ldots, n\}\}$. Since our polynomial f is usually fixed, we will often denote $\Omega(f)$ and $\Delta(f)$ just by Ω and Δ for short.

Let $\widetilde{f}(\underline{X}, Y) = Y^{2d} f(\frac{X_1}{Y}, \ldots, \frac{X_n}{Y})$. From (3), it is clear that

$$\widetilde{f}(\underline{X}, Y) = f_0 Y^{2d} + \sum_{\alpha \in \Omega} f_\alpha \underline{X}^\alpha Y^{2d - |\alpha|} + \sum_{i=1}^n f_{2d,i} X_i^{2d}$$

is a form of degree $2d$, called the homogenization of f. We have the following well-known result:

Proposition 1.1. *f is sos if and only if $\widetilde{f}$ is sos.*

Proof. See [7, Prop. 1.2.4]. □

For a (univariate) polynomial of the form $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero, we denote by $C(p)$ the unique positive root of p [10, Th. 1.1.3]. For any polynomial $q(t) = \sum_{i=0}^{n} b_i t^i$, $b_n \neq 0$, the roots of q are bounded in absolute value by $C(t^n - \sum_{i=0}^{n-1} \frac{|b_i|}{|b_n|} t^i)$. By convention, $C(t^n) := 0$.

There are various upper bounds for $C(p)$ which are expressible in an elementary way in terms of the coefficients of p, for example,

Proposition 1.2. *Suppose $p(t) = t^n - \sum_{i=0}^{n-1} a_i t^i$, where each a_i is nonnegative and at least one a_i is nonzero. Then*

(1) $C(p) \leq \max\{1, a_0 + a_1 + \cdots + a_{n-1}\}$,
(2) $C(p) \leq \max\{a_0, 1 + a_1, 1 + a_2, \ldots, 1 + a_{n-1}\}$,
(3) $C(p) \leq 2\max\{a_{n-1}, (a_{n-2})^{1/2}, (a_{n-3})^{1/3}, \ldots, (a_0)^{1/n}\}$.

Proof. Bounds (1) and (2) are due basically to Cauchy. See [1] for these bounds and for other bounds of this sort. See [4, Ex. 4.6.2: 20] for bound (3). □

2. Application of Lasserre's Result

Lasserre proved in [6] that for a given $f \in \mathbb{R}[\underline{X}]$ of degree $2d$, if

(L1) $\quad f_0 \geq \sum_{\alpha \in \Delta} |f_\alpha|$ and

(L2) $\quad \min_{i=1,\ldots,n} f_{2d,i} \geq \sum_{\alpha \in \Delta} |f_\alpha| \frac{|\alpha|}{2d}$

then $f \in \sum \mathbb{R}[\underline{X}]^2$; see [6, Th. 3].

Theorem 2.1. $X_1^{2d} + \cdots + X_n^{2d} \in \Sigma_{2d,n}^{\circ}$.

Th. 2.1 is proved already in [8, Prop. 5.3(2)]. We give another proof here which is based on Lasserre's result. Th. 2.1 can also be deduced from Fidalgo-Kovacec's result [2, Th. 4.3]. See Section 3 below for the statement of [2, Th. 4.3].

Proof. Let $p(\underline{X}) = X_1^{2d} + \cdots + X_n^{2d} + h(\underline{X})$ where $h(\underline{X})$ is any form of degree $2d$ whose coefficients have absolute value $\leq \epsilon$ where ϵ is some small positive real. One sees that, for ϵ sufficiently small, the polynomial $q \in \mathbb{R}[X_1, \ldots, X_{n-1}]$ defined by $q(X_1, \ldots, X_{n-1}) = p(X_1, \ldots, X_{n-1}, 1)$ satisfies (L1) and (L2) so, by Lasserre's result, q is sos. Then, by Prop. 1.1, the homogenization of q (which is p) belongs to $\Sigma_{2d,n}$. □

Remark 2.2. For any cone C in a finite dimensional real vector space V, if $f \in C^\circ$ and $g \in V$ then $g \in C^\circ$ iff $g - \epsilon f \in C$ for some real $\epsilon > 0$.

Proof. Suppose $g - \epsilon f \in C$. Let $h \in V$. Since f belongs to the interior of C, there exists some real $\delta > 0$ such that $f + \frac{\delta}{\epsilon} h \in C$. Then $g + \delta h = (g - \epsilon f) + \epsilon(f + \frac{\delta}{\epsilon} h) \in C$. This proves that g belongs to the interior of C. The other implication is clear. □

Every polynomial f of degree $2d$ decomposes as $f = f_0 + f_1 + \cdots + f_{2d}$ (the homogeneous decomposition of f) where each f_i is a form of degree i. The following theorem gives a sufficient condition for $f_{sos} \neq -\infty$ in terms of the highest degree form, f_{2d}, and a concrete lower bound for f_{sos}:

Theorem 2.3. *If $f_{2d} \in \Sigma^\circ_{2d,n}$ then $f_{sos} \geq r_L$, where*

$$r_L := f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|}, \ k := C(t^{2d} - \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha| \frac{|\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|})$$

and $\epsilon > 0$ is given such that $f_{2d} - \epsilon(\sum_{i=1}^n X_i^{2d}) \in \Sigma_{2d,n}$.

Note: (i) If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $k = C(t^{2d}) := 0$ and $r_L := f_0$. (ii) Th. 2.3 proves in particular that if $f_{2d} \in \Sigma^\circ_{2d,n}$ then $f_{sos} \neq -\infty$, i.e., it provides another proof of [8, Prop. 5.1]. (iii) If $\ell \geq k$ then

$$f_0 - \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha|\epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|} \leq f_0 - \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha|\epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|} = r_L.$$

In this way, by taking ℓ to be one of the upper bounds for k mentioned in Prop. 1.2, we obtain a lower bound $f_0 - \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha|\epsilon^{-\frac{|\alpha|}{2d}} \ell^{|\alpha|}$ for f_{sos} which is expressible in an elementary way in terms of ϵ and the coefficients f_α, $|\alpha| < 2d$. (iv) We are assuming here that ϵ is given. Refer to Th. 4.2 below for a method for computing ϵ which applies in certain cases.

Proof. Since $f_{2d} \in \Sigma^\circ_{2d,n}$, by Th. 2.1 and Rem. 2.2, there exists $\epsilon > 0$ such that $f_{2d} = \epsilon(X_1^{2d} + \cdots + X_n^{2d}) + g$ for some $g \in \Sigma_{2d,n}$. Scaling suitably ($X_i \mapsto \frac{X_i}{\sqrt[2d]{\epsilon}}$), we can assume that $\epsilon = 1$. Let $\hat{f} := f - g$. Decomposing $\hat{f}$ as in equation (3) yields

$$\hat{f} = f_0 + \sum_{\alpha\in\Omega,|\alpha|<2d} f_\alpha \underline{X}^\alpha + \sum_{i=1}^n X_i^{2d}. \tag{4}$$

If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $\hat{f} - r_L = \hat{f} - f_0$ is sos, using equation (4) and the definition of Δ, so $f - r_L$ is also sos and the result is clear. Thus we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Scaling by $X_i \mapsto kX_i$, and rewriting condition (L2) for the polynomial $\hat{f}(k\underline{X}) - r$, using equation (4), yields

$$k^{2d} \geq \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha| \frac{|\alpha|}{2d} k^{|\alpha|}.$$

By definition of k, $k^{2d} = \sum_{\alpha\in\Delta,|\alpha|<2d} |f_\alpha| \frac{|\alpha|}{2d} k^{|\alpha|}$, so condition (L2) holds for $\hat{f}(k\underline{X}) - r$. Rewriting (L1) for the polynomial $\hat{f}(k\underline{X}) - r$, we see that if $r \leq r_L$ then (L1) holds for $\hat{f}(k\underline{X}) - r$ so $\hat{f} - r$ is sos and hence also $f - r$ is sos. □

3. Application of Fidalgo-Kovacec's Results

Fidalgo and Kovacec in [2] proved that if $f \in \mathbb{R}[\underline{X}]$ is a form of degree $2d$ and

$$\text{(FK)} \quad \min_{i=1,\ldots,n} f_{2d,i} \geq \tfrac{1}{n}(\tfrac{n}{2d})^{2d} \sum_{\alpha\in\Delta} |f_\alpha|\alpha^\alpha$$

then f is a sum of squares [2, Th. 4.3]. Here, $\alpha^\alpha := \alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}$ (the convention being that $0^0 := 1$).

If $f \in \mathbb{R}[\underline{X}]$ is arbitrary of degree $2d$ then, rewriting (FK) for $\widetilde{f}$ (the homogenization of f), we see that if

$$\text{(FK}'\text{)} \quad \min_{i=1,\ldots,n} \{f_{2d,i}, f_0\} \geq \tfrac{1}{n+1}(\tfrac{n+1}{2d})^{2d} \sum_{\alpha\in\Delta} |f_\alpha|\alpha^\alpha(2d - |\alpha|)^{2d-|\alpha|}$$

then f is sos, by Prop. 1.1. So $f-r$ is sos as long as $\min_{i=1,\cdots,n}\{f_{2d,i}, f_0-r\} \geq$ the right side of (FK′). One can use more or less the same idea used in the proof of Th. 2.3 to find a lower bound for f_{sos} using (FK′) as follows:

Theorem 3.1. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then $f_{sos} \geq r_{FK}$, where $r_{FK} := f_0 - k^{2d}$, $k := C(t^{2d} - \sum_{i=1}^{2d-1} b_i t^i)$,*

$$b_i := \frac{1}{n+1}\left(\frac{n+1}{2d}\right)^{2d}(2d-i)^{2d-i}\epsilon^{-\frac{i}{2d}}\sum_{\alpha\in\Delta,|\alpha|=i}|f_\alpha|\alpha^\alpha,\ i=1,\ldots,2d-1$$

and $\epsilon > 0$ is given as in Th. 2.3.

Note: If $\ell \geq k$ then

$$f_0 - \sum_{i=1}^{2d-1} b_i\ell^i \leq f_0 - \sum_{i=1}^{2d-1} b_i k^i = f_0 - k^{2d} = r_{FK}$$

so, using Prop. 1.2 again, we get another lower bound for f expressible in an elementary way in terms of ϵ and the coefficients f_α, $|\alpha| < 2d$.

Proof. After scaling we can assume that $\epsilon = 1$ and $f_{2d} = X_1^{2d} + \cdots + X_n^{2d} + g$, where $g \in \Sigma_{2d,n}$. If $\{\alpha \in \Delta \mid |\alpha| < 2d\} = \emptyset$, then $b_i = 0$ for $i = 1, \ldots 2d-1$, $k = 0$ (by definition of $C(t^{2d})$), so $r_{FK} = f_0$. In this case the result is clear. So we can assume $\{\alpha \in \Delta \mid |\alpha| < 2d\} \neq \emptyset$, so $k > 0$. Set $r = r_{FK}$. Rewriting condition (FK′) for $\hat{f}(k\underline{X}) - r$, where $\hat{f} := f - g$, yields the condition:

$$\min\{(f_0-r), k^{2d}\} \geq \sum_{i=1}^{2d-1} b_i k^i. \tag{5}$$

By definition of k and r, (5) holds, in fact, $f_0 - r = k^{2d} = \sum_{i=1}^{2d-1} b_i k^i$. This proves that $\hat{f} - r$ is sos and hence also that $f - r$ is sos. □

According to [2, Th. 2.3], a homogeneous polynomial $\sum_{i=1}^{n}\beta_i X_i^{2d} - \mu\underline{X}^\alpha$, such that $\alpha_i > 0$ and $\beta_i \geq 0$ for every $i = 1, \ldots, n$ and $\mu \geq 0$ if all α_i are even, is sos if and only if

$$|\mu| \leq 2d\prod_{i=1}^{n}\left(\frac{\beta_i}{\alpha_i}\right)^{\frac{\alpha_i}{2d}}. \tag{6}$$

Theorem 3.2. *If $f_{2d} \in \Sigma^{\circ}_{2d,n}$ then*

$$f_{sos} \geq r_{dmt} := f_0 - \sum_{\alpha\in\Delta,|\alpha|<2d}(2d-|\alpha|)\left[\left(\frac{f_\alpha}{2d}\right)^{2d}\left(\left(\frac{t}{\epsilon}\right)^{|\alpha|}\alpha^\alpha\right)\right]^{\frac{1}{2d-|\alpha|}},$$

where $t := |\{\alpha \in \Delta \mid |\alpha| < 2d\}|$ and $\epsilon > 0$ is given as in Th. 2.3.

Proof. Let $\Delta' = \{\alpha \in \Delta \mid |\alpha| < 2d\}$. After scaling, we can assume that $\epsilon = 1$. Let $\overline{f} = f_0 + \sum_{\alpha\in\Delta'} f_\alpha\underline{X}^\alpha + X_1^{2d} + \cdots + X_n^{2d}$ and let $F(\underline{X}, Y)$ denote the homogenization of $\overline{f}(\sqrt[2d]{t}\underline{X}) - r$, where $r := f_0 - \sum_{\alpha\in\Delta'} r_\alpha$, each $r_\alpha \geq 0$. Then

$$\begin{aligned} F(\underline{X},Y) &= (f_0-r)Y^{2d} + \sum_{\alpha\in\Delta'}(X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d}\underline{X}^\alpha Y^{2d-|\alpha|}) \\ &= \sum_{\alpha\in\Delta'}(r_\alpha Y^{2d} + X_1^{2d} + \cdots + X_n^{2d} + f_\alpha t^{|\alpha|/2d}\underline{X}^\alpha Y^{2d-|\alpha|}). \end{aligned}$$

By (6), each term appearing in this sum will be sos if

$$|f_\alpha| t^{\frac{|\alpha|}{2d}} \le 2d \left(\frac{r_\alpha}{2d-|\alpha|}\right)^{\frac{2d-|\alpha|}{2d}} \prod_{\alpha_i\neq 0} \left(\frac{1}{\alpha_i}\right)^{\frac{\alpha_i}{2d}},$$

or, equivalently, if

$$r_\alpha \ge (2d-|\alpha|)\left[\left(\frac{f_\alpha}{2d}\right)^{2d} t^{|\alpha|}\alpha^\alpha\right]^{\frac{1}{2d-|\alpha|}}.$$

Hence if $r \le r_{dmt}$ then $\overline{f} - r$ is sos, so also $f - r$ is sos. □

4. Further remarks

(1) The lower bounds r_L, r_{FK} and r_{dmt} for f_{sos} described in Ths. 2.3, 3.1 and 3.2 are not comparable.

Example 4.1.

(a) For $f(X,Y) = X^6 + Y^6 + 2X^3Y^2 + 1$, we have $r_L = 1 - 2(\frac{5}{3})^5 \approx -24.7202$, $r_{FK} = 1 - (\frac{9}{8})^6 \approx -1.0273$ and $r_{dmt} = \frac{23}{27} \approx 0.8519$, so r_{dmt} is best.

(b) For $f(X) = X^6 - X^4 - X^2$, $r_L = -2$, $r_{FK} \approx -3.9980$ and $r_{dmt} \approx -2.1322$, so r_L is best.

(c) For $f(X,Y,Z) = X^4 + Y^4 + Z^4 + aXYZ + bXY$, $a, b > 0$, $r_{FK} > r_{dmt} > r_L$, so r_{FK} is best, provided b is sufficiently small compared to a.

(2) To be able to compute r_L, r_{FK} and r_{dmt} one needs to know ϵ and the coefficients f_α, $|\alpha| < 2d$. What can one do if ϵ is not given, i.e., if only the coefficients f_α, $|\alpha| \le 2d$ are given? The results in [2, Th. 4.3] and [6, Th. 3] provide sufficient conditions for $f_{2d} \in \Sigma^\circ_{2d,n}$ to hold and have the nice additional property of allowing computation of ϵ:

Theorem 4.2. *Suppose* $\epsilon := \max\{\epsilon_1, \epsilon_2\} > 0$ *where*

$$\epsilon_1 := \min_{i=1,\dots,n} f_{2d,i} - \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha|,\ \ \epsilon_2 := \min_{i=1,\dots,n} f_{2d,i} - \frac{1}{n}\left(\frac{n}{2d}\right)^{2d} \sum_{\alpha\in\Delta,|\alpha|=2d} |f_\alpha|\alpha^\alpha.$$

Then $f_{2d} \in \Sigma^\circ_{2d,n}$ *and* $f_{2d} - \epsilon\sum_{i=1}^n X_i^{2d} \in \Sigma_{2d,n}$.

Proof. Suppose $\epsilon = \epsilon_2$. The inequality (FK) holds for the form $f_{2d} - \epsilon\sum_{i=1}^n X_i^{2d}$, by definition of ϵ_2, so $f_{2d} - \epsilon\sum_{i=1}^n X_i^{2d}$ is sos and $f_{2d} \in \Sigma^\circ_{2d,n}$ by Rem. 2.2. Suppose $\epsilon = \epsilon_1$. Consider the dehomogenization $h(X_1,\dots,X_{n-1}) = f_{2d}(X_1,\dots,X_{n-1},1) - \epsilon(\sum_{i=1}^{n-1} X_i^{2d} + 1)$ of $f_{2d} - \epsilon\sum_{i=1}^n X_i^{2d}$. The inequality (L1) holds for h by definition of ϵ_1. The inequality (L2) holds using

$$\min_{i=1,\dots,n-1}(f_{2d,i} - \epsilon_1) \ge \sum_{\alpha\in\Delta,|\alpha|=2d} |f_\alpha| \ge \sum_{\alpha\in\Delta,|\alpha|=2d} |f_\alpha|\frac{|\alpha| - \alpha_n}{2d}.$$

Thus h is sos. Rehomogenizing and applying Prop. 1.1 this implies that $f_{2d} - \epsilon\sum_{i=1}^n X_i^{2d}$ is sos. □

In [6, Cor. 5] it is proved that if $\epsilon_1 > 0$ then $f_{sos} \neq -\infty$. Using Th. 4.2 we see that this can also be deduced from Th. 2.3. We remark also that, in the statement of Th. 4.2, it is possible to improve slightly on the definition of ϵ_1. Namely, one can take

$$\epsilon_1 := \max\{\epsilon(1), \dots, \epsilon(n)\}$$

where

$$\epsilon(i) := \min\{f_{2d,i} - \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha|,\ \min_{j\neq i} f_{2d,j} - \sum_{\alpha\in\Delta, |\alpha|=2d} |f_\alpha| \frac{|\alpha| - \alpha_i}{2d}\}.$$

This is more or less clear from the proof of Th. 4.2. If $\epsilon = \epsilon_1 = \epsilon(i)$ just dehomogenize $f_{2d} - \epsilon \sum_{j=1}^n X_j^{2d}$ with respect to the variable X_i and apply Lasserre's [6, Th. 3] to deduce that $f_{2d} - \epsilon \sum_{j=1}^n X_j^{2d}$ is sos.

(3) So far we have been assuming that $f_{2d} \in \Sigma^\circ_{2d,n}$ and we have used this assumption to determine lower bounds for f_{sos}. What can one say if one assumes only that $f_{2d} \in P^\circ_{2d,n}$? Suppose $\epsilon > 0$ is given such that $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in P_{2d,n}$. One can then define r_L exactly as in Th. 2.3, but using this new ϵ, i.e.,

$$r_L := f_0 - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \epsilon^{-\frac{|\alpha|}{2d}} k^{|\alpha|},\ k := C(t^{2d} - \sum_{\alpha\in\Delta, |\alpha|<2d} |f_\alpha| \frac{|\alpha|}{2d} \epsilon^{-\frac{|\alpha|}{2d}} t^{|\alpha|}).$$

The r_L defined in this way might not be a lower bound for f_{sos} (it is even possible that $f_{sos} = -\infty$), but it will be a lower bound for f_*. Similar remarks apply to the other bounds r_{FK} and r_{dmt}.

Theorem 4.3. *If $f_{2d} \in P^\circ_{2d,n}$ and $\epsilon > 0$ is such that $f_{2d} - \epsilon \sum_{i=1}^n X_i^{2d} \in P_{2d,n}$ then r_L, r_{FK} and r_{dmt}, defined as in Ths. 2.3, 3.1 and 3.2, respectively, but using this new choice of ϵ, are lower bounds for f on $\mathbb{R}^n$.*

Proof. Argue as in the proof of Ths. 2.3, 3.1 and 3.2. The form g is no longer sos but it is positive semidefinite, which is all one needs for the conclusion. □

Note: In Th. 4.3, the largest possible choice for ϵ is the minimum value of the rational function $f_{2d}/\sum_{i=1}^n X_i^{2d}$ on the $n-1$-sphere

$$\mathbb{S}^{n-1} := \{\underline{a} \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 = 1\}.$$

(4) We know that for any $p \in P^\circ_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_* \neq -\infty$ and, for any $p \in \Sigma^\circ_{2d,n}$ and any $g \in \mathbb{R}[\underline{X}]_{2d-1}$, $(p+g)_{sos} \neq -\infty$. Note that if $p \in P_{2d,n}$ is not positive definite then there exists $\underline{0} \neq \underline{a} \in \mathbb{R}^n$ such that $p(\underline{a}) = 0$. Let $g(\underline{X}) = \sum_{i=1}^n a_i X_i$. Then $(p+g)(t\underline{a}) = t\|\underline{a}\|^2 \to -\infty$ as $t \to -\infty$, so $(p+g)_* = -\infty$. Therefore for any $p \in \partial P_{2d,n}$ ($\partial P_{2d,n}$ denotes the boundary of $P_{2d,n}$, i.e. $\partial P_{2d,n} = P_{2d,n} \setminus P^\circ_{2d,n}$), there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$, such that $(p+g)_* = -\infty$. The validity of the corresponding result for boundary points of $\Sigma_{2d,n}$ is unknown to the authors.

Question 4.4. Is it true that for any $p \in \partial\Sigma_{2d,n}$ there exists $g \in \mathbb{R}[\underline{X}]_{2d-1}$ such that $(p+g)_{sos} = -\infty$?

The answer to this question is 'yes' if $n \leq 2$ or $d = 1$ or ($n = 3$ and $d = 2$) by Hilbert's result [3]. In fact these are precisely the cases where $P_{2d,n}$ and $\Sigma_{2d,n}$ coincide.

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Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, SK S7N 5E6, Canada
E-mail address: mehdi.ghasemi@usask.ca, marshall@math.usask.ca