# LOCAL-GLOBAL PROPERTIES OF POSITIVE PRIMITIVE FORMULAS IN THE THEORY OF SPACES OF ORDERINGS

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The main examples driving the theory of spaces of orderings and reduced special groups are the spaces of orderings arising in the study of formally real fields [14]. These have application to the study of semialgebraic sets and semianalytic sets in real algebraic geometry and in real analytic geometry [1] [3] [18].

At the same time, spaces of orderings occur naturally in a wide variety of contexts: semilocal rings [13] (more generally, rings with many units [22]) or skewfields [9] (more generally, domains [15] or ternary fields [12]) or $*$–fields with involution [10] (more generally, $*$–domains [19]). Spaces of orderings are the 'local objects' in the 'global' theory of spaces of signs (abstract real spectra) developed in [1] [18].

We consider pp (positive primitive) formulas in the language of reduced special groups as in [20]. We denote the pp formula

$$\exists t_1 \dots \exists t_n \ P(t_1, \dots, t_n)$$

by $\exists \underline{t} P(\underline{t})$ for short, where $\underline{t} = (t_1, \dots, t_n)$. $P(\underline{t})$ is a finite conjunction of atomic formulas. The atomic formulas, by definition, have the form '$a(\underline{t}) \in D\langle b(\underline{t}), c(\underline{t})\rangle$' where each of $a(\underline{t})$, $b(\underline{t})$, $c(\underline{t})$ is some finite product of the $t_1, \dots, t_n$ times $\pm 1$ times a parameter. In [20], the following basic question is asked:

**Question 1.** *For a space of orderings $(X, G)$ and a pp formula $\exists \underline{t} P(\underline{t})$ with parameters in $G$ is it true that if the formula $\exists \underline{t} P(\underline{t})$ holds in every finite subspace of $(X, G)$ then it holds in $(X, G)$?*

The significance of Question 1 stems from the fact that many open questions concerning spaces of orderings which are easily settled in the finite case using the Structure Theorem (see Section 1) can be phrased as pp formulas.

In [20] it is shown that the class of spaces of orderings for which the answer to Question 1 is 'yes' (for every pp formula) contains all spaces of orderings of stability index 1 and is closed under the operations of direct sum and group extension. This includes spaces of orderings of fields of transcendence degree 1 over a real closed field. It seems unlikely that the answer is 'yes' in general. In fact, if this was actually the case, then a number of hard open problems would be solved; see [20].

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At the same time, there are key results in the theory that assert that the answer to Question 1 is 'yes' for certain sorts of pp formula. The pp formula '$\phi$ is isotropic', $\phi$ a form with entries in $G$, has this property [16, Th. 1.4], as does the pp formula '$\cap_{i=1}^{n} D(\phi_i) \neq \emptyset$', $\phi_i$ a form with entries in $G$, [17, Th. 2.1] (for any space of orderings $(X, G)$). This suggests that although the answer to Question 1 is almost certainly 'no' in general, it is still important to know more about the sorts of pp formulas for which the answer is 'yes'.

Section 1 is introductory in nature and fixes notation. The experts may scan this section briefly or omit it entirely. In Section 2 we produce a large class of pp formulas which for which the answer to Question 1 is 'yes', at least, if $(X, G)$ has finite stability index. This greatly extends the list of known examples. The proof is a modification of the proof of the results in [16] and [17] referred to above. In Section 3 we consider pp formulas of the special type '$b \in \prod_{i=1}^{n} D\langle 1, a_i\rangle$', $b, a_1, \dots, a_n \in G$. The simplest sorts of pp formula for which the answer to Question 1 is open are of this type with $n = 3$. Consequently, we are interested in the following special case of Question 1:

**Question 2.** *For a space of orderings $(X, G)$ and elements $a_1, \dots, a_n, b \in G$ is it true that if the formula $b \in \prod_{i=1}^{n} D\langle 1, a_i\rangle$ holds in every finite subspace of $(X, G)$ then it holds in $(X, G)$?*

In Section 4 we show that, for the space of orderings of the function field in two variables $x, y$ over a real closed field, and the pp formula $f \in D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle$, the answer to Question 2 is 'yes'.

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## 1. Introduction

Suppose $(X, G)$ is a space of orderings. There are several equivalent definitions; see [1] [11] [16] [18]. $X$ is a non-empty set. $G$ is a group of exponent 2. The image of $(\sigma, a)$, $\sigma \in X$, $a \in G$, under the basic pairing $X \times G \to \{-1, 1\}$ will be denoted by $\sigma a$. Thus

$$\begin{cases} \sigma(ab) = (\sigma a)(\sigma b), \\ \sigma a = 1 \ \forall\ \sigma \in X \ \Rightarrow \ a = 1, \text{ and} \\ \sigma a = \tau a \ \forall\ a \in G \ \Rightarrow \ \sigma = \tau. \end{cases}$$

For the non-expert it is best to think concretely in terms of the space of orderings of a formally real field $F$. Here, $X$ is the set of all orderings of $F$, i.e., subsets $\sigma$ of $F$ closed under addition and multiplication and satisfying $\sigma \cup -\sigma = F$ and $\sigma \cap -\sigma = \{0\}$, $G$ is the factor group $F^{\cdot}/\Sigma F^{2\,\cdot}$ of the multiplicative group $F^{\cdot}$ of $F$, where $\Sigma F^2$ denotes the set of sums of squares in $F$, and the pairing $X \times G \to \{-1, 1\}$ is given by

$$\sigma\left(r\Sigma F^{2\,\cdot}\right) = \begin{cases} 1 & \text{if } r \in \sigma \\ -1 & \text{if } r \in -\sigma \end{cases}.$$

$G$ has a distinguished element $-1$ satisfying $\sigma(-1) = -1$ $\forall \sigma \in X$. In the field example $-1$ is just the coset modulo $\Sigma F^{2\,\cdot}$ of the element $-1 \in F^{\cdot}$. For $a \in G$, define $\widehat{a} : X \to \{-1, 1\}$ by $\widehat{a}(\sigma) = \sigma a$. The map $a \mapsto \widehat{a}$ identifies $G$ with the subgroup $\widehat{G} = \{\widehat{a} \mid a \in G\}$ of the group of all functions from $X$ into the group $\{-1, 1\}$. $\widehat{-1}$ is the constant function $-1$ on $X$. $X$ is given the weakest topology such that each $\widehat{a}$, $a \in G$, is continuous. The sets $\mathrm{U}(a) := \{\sigma \in X \mid \sigma a = 1\}$, $a \in G$, form a subbasis of open sets. With this topology $X$ is compact, Hausdorff and totally disconnected. There is a relation $\cong$ (called *isometry*) defined on pairs of elements $\langle a, b\rangle$, $a, b \in G$, defined by

$$\langle a, b\rangle \cong \langle c, d\rangle \text{ iff } \forall \sigma \in X,\ \sigma a + \sigma b = \sigma c + \sigma d \text{ (addition in } \mathbb{Z}).$$

$(G, -1, \cong)$ is the (reduced) *special group* associated to the space of orderings $(X, G)$, terminology as in [11]. If $(X, G)$ is the singleton space of orderings, i.e., $|X| = 1$, then $G = \{-1, 1\}$. For $\sigma \in X$, denote by $\widehat{\sigma} : G \to \{-1, 1\}$ the function defined by $\widehat{\sigma}(a) = \sigma a$. The map $\sigma \mapsto \widehat{\sigma}$ defines a one-to-one correspondence (even a homeomorphism) between $X$ and the subset $\widehat{X} = \{\widehat{\sigma} \mid \sigma \in X\}$ of the character group of $G$. $\widehat{X}$ is precisely the set of all special group homomorphisms from $G$ to the two-element (reduced) special group $\{-1, 1\}$. It follows that the space of orderings $(X, G)$ can be recovered from the special group $(G, -1, \cong)$.

The *value set* of $\langle a, b\rangle$, $a, b \in G$, is

$$D\langle a, b\rangle = \{c \in G \mid \exists d \in G \text{ such that } \langle a, b\rangle \cong \langle c, d\rangle\}.$$

In the field example, if $a = r_1 \Sigma F^{2\,\cdot}$, $b = r_2 \Sigma F^{2\,\cdot}$, then

$$D\langle a, b\rangle = \{\ (r_1 s_1 + r_2 s_2)\Sigma F^{2\,\cdot} \mid s_1, s_2 \in \Sigma F^2, r_1 s_1 + r_2 s_2 \neq 0\}.$$

Isometry can be recovered from the ternary relation '$c \in D\langle a, b\rangle$' on $G$ via

$$\langle a, b\rangle \cong \langle c, d\rangle \text{ iff } c \in D\langle a, b\rangle \text{ and } ab = cd.$$

Equality in $G$ can be recovered from this relation via $a = 1$ iff $a \in D\langle 1, 1\rangle$.

Subspaces $(Y, H)$ of a space of orderings $(X, G)$ [1] [18] correspond to special group homomorphisms $\pi : G \to H$ which are surjective and have the property that every special group homomorphism from $G$ to the two-element special group $\{-1, 1\}$ which is trivial on the kernel of $\pi$ factors through $H$.

In the field example, subspaces correspond to (proper) preorderings of $F$, i.e., subsets $T$ of $F$ which are closed under addition and multiplication and satisfy $r^2 \in T$ for all $r \in F$ and $-1 \notin T$. The subspace of the space of orderings of $F$ corresponding to the preordering $T$ of $F$ is $(Y, H)$ where $Y$ is the set of orderings $\sigma$ of $F$ such that $T \subseteq \sigma$ and $H = F^{\cdot}/T^{\cdot}$. The basic pairing $Y \times H \to \{-1, 1\}$ is the obvious one:

$$\sigma\left(rT^{\cdot}\right) = \begin{cases} 1 & \text{if } r \in \sigma \\ -1 & \text{if } r \in -\sigma \end{cases}.$$

The distinguished element $-1$ is the coset modulo $T^{\cdot}$ of $-1 \in F^{\cdot}$. For $a = r_1 T^{\cdot}$, $b = r_2 T^{\cdot}$, the value set $D\langle a, b\rangle$ consists of all elements of the form $(r_1t_1 + r_2t_2)T^{\cdot}$ with $t_1, t_2 \in T$, $r_1t_1 + r_2t_2 \neq 0$.

Deeper results in the theory typically involve an analysis of quadratic forms. Let $(X, G)$ be a space of orderings. A (quadratic) *form* of *dimension* $n \geq 1$ with entries in $G$ is just an $n$-tuple $\phi = \langle a_1, \dots, a_n\rangle$, $a_1, \dots, a_n \in G$. The *signature* of $\phi$ at $\sigma \in X$ is $\phi(\sigma) := \sum_{i=1}^{n} \sigma a_i$ (viewed as an element of $\mathbb{Z}$). Isometry and value sets of 2-dimensional forms have already been defined. In general, *isometry* of $n$-dimensional forms is defined by

$$\phi \cong \psi \;\Leftrightarrow\; \dim(\phi) = \dim(\psi) \text{ and } \forall \sigma \in X \;\phi(\sigma) = \psi(\sigma)$$

and the *value set* of $\phi = \langle a_1, \dots, a_n\rangle$ is defined by

$$D(\phi) = \{b_1 \in G \mid \exists b_2, \dots, b_n \in G \text{ such that } \phi \cong \langle b_1, \dots, b_n\rangle\}.$$

Isometry and value sets are also described inductively for $n \geq 3$ by

$$\langle a_1, \dots, a_n\rangle \cong \langle b_1, \dots, b_n\rangle \text{ iff } \exists a, b, c_3, \dots, c_n \in G \text{ such that}$$
$$\langle a_1, a\rangle \cong \langle b_1, b\rangle,\; \langle a_2, \dots, a_n\rangle \cong \langle a, c_3, \dots, c_n\rangle \text{ and } \langle b_2, \dots, b_n\rangle \cong \langle b, c_3, \dots, c_n\rangle,$$

and

$$D\langle a_1, \dots, a_n\rangle = \{b \in G \mid b \in D\langle a_1, c\rangle \text{ for some } c \in D\langle a_2, \dots, a_n\rangle\};$$

see [18, Th. 2.2.3]. If $(X, G)$ is the space of orderings of a formally real field $F$, and $a_i = r_i \Sigma F^{2\,\cdot}$, $i = 1, \dots, n$, $n \geq 1$, then

$$D\langle a_1, \dots, a_n\rangle = \{(\sum_{i=1}^{n} r_i s_i)\Sigma F^{2\,\cdot} \mid s_1, \dots, s_n \in \Sigma F^2,\; \sum_{i=1}^{n} r_i s_i \neq 0\}.$$

For the subspace $(Y, H)$ of $(X, G)$ corresponding to a (proper) preordering $T$ of $F$ one has an analogous description of value sets, but with $\Sigma F^2$ replaced by $T$.

A form $\phi$ of dimension $\geq 2$ with entries in $G$ is *isotropic* if there exist $b_3, \dots, b_n \in G$ such that $\phi \cong \langle 1, -1, b_3, \dots, b_n\rangle$. Equivalently, $\langle a_1, \dots, a_n\rangle$ is isotropic iff $-a_1 \in D\langle a_2, \dots, a_n\rangle$. Other descriptions are given in [18, Th. 2.2.6]. For $(X, G)$ the space of orderings of a formally real field $F$, the form $\langle a_1, \dots, a_n\rangle$ where $a_i = r_i \Sigma F^{2\,\cdot}$, $i = 1, \dots, n$, is isotropic iff there exist $s_1, \dots, s_n \in \Sigma F^2$ not all zero such that $\sum_{i=1}^{n} r_i s_i = 0$. For the subspace $(Y, H)$ of $(X, G)$ corresponding to a (proper) preordering $T$ of $F$ one has an analogous description of isotropy, but with $\Sigma F^2$ replaced by $T$.

We refer the reader to [16, Th. 1.4] for the statement and proof of the Isotropy Theorem. The Isotropy Theorem was proved initially in the field case; see [4] [21]. See [17, Th. 2.1] for an extension of the Isotropy Theorem.

Given two spaces of orderings $(X_i, G_i)$, $i = 1, 2$, one can form the *direct sum* $(X_1, G_1) \oplus (X_2, G_2)$ which is again a space of orderings. If we denote the direct sum by $(X, G)$, then $G$ is the product of the groups $G_1$ and $G_2$ and $X$ is the disjoint union of the sets $X_1$ and $X_2$. The basic pairing $X \times G \to \{-1, 1\}$ is defined by

$$\sigma(a_1, a_2) = \begin{cases} \sigma a_1 \text{ if } \sigma \in X_1 \\ \sigma a_2 \text{ if } \sigma \in X_2 \end{cases}.$$

As a special group, $G$ is the product (in the category of special groups) of $G_1$ and $G_2$.

Given a space of orderings $(\overline{X}, \overline{G})$ and a group $\Delta$ of exponent 2, one can form the *group extension* of $(\overline{X}, \overline{G})$ by $\Delta$ which is again a space of orderings. If we denote this space of orderings by $(X, G)$, then $G$ is the product of the groups $\overline{G}$ and $\Delta$ and $X$ is the product of the sets $\overline{X}$ and $\chi(\Delta)$, where $\chi(\Delta)$ denotes the character group of $\Delta$. The basic pairing $X \times G \to \{-1, 1\}$ is given by $(\sigma, \delta)(a, d) = (\sigma a)(\delta(d))$.

A space of orderings $(X, G)$ is called a *fan* if it is a group extension of the singleton space or, equivalently, if $\widehat{X} := \{\widehat{\sigma} \mid \sigma \in X\}$ is the set of *all* group homomorphisms from $G$ to $\{-1, 1\}$ sending $-1$ to $-1$. In the field case fans and group extensions are closely related to real valuations, e.g., see [5] and [16, Th. 2.8]. If $(X, G)$ is a finite fan then $|X| = 2^k$ and $|G| = 2^{k+1}$ for some integer $k \geq 0$. The *stability index* of a space of orderings $(X, G)$, denoted $\operatorname{stab}(X, G)$, is the maximum $k$ (or $\infty$) such that $(X, G)$ has a subspace $(Y, H)$ which is a fan with $|Y| = 2^k$. See [18, Th. 3.6.3] for other characterizations of the stability index.

The *chain length* $\operatorname{cl}(X, G)$ of a space of orderings $(X, G)$ is the maximum integer $k$ (or $\infty$) such that there exists $a_i \in G$, $i = 0, \dots, k$ such that $D\langle 1, a_{i-1}\rangle \subsetneqq D\langle 1, a_i\rangle$ for $i = 1, \dots, n$. Fans have chain length $\leq 2$. If $(X, G)$ is generated by a finite number of subspaces $(Y_i, H_i)$, $i = 1, \dots, n$, then $\operatorname{cl}(X, G) \leq \sum_{i=1}^{n} \operatorname{cl}(Y_i, H_i)$.

**Structure Theorem.** *For a space of orderings $(X, G)$, the following are equivalent:*

1. *$(X, G)$ has finite chain length.*
2. *$(X, G)$ is generated by finitely many subspaces $(Y_i, H_i)$, $i = 1, \dots, n$, which are fans.*
3. *There exist finitely many subspaces $(Y_i, H_i)$, $i = 1, \dots, n$ of $(X, G)$ which are fans such that $X = \cup_{i=1}^{n} Y_i$.*
4. *$(X, G)$ is built up in a finite number of steps starting from the singleton space, by forming direct sums and group extensions.*

The Structure Theorem was proved first in the field case using valuation theory [7] [8]. A bit later a proof was discovered which avoids valuation theory and is valid for any space of orderings [16, Th. 1.6].

## 2. A CLASS OF PP FORMULAS FOR WHICH THE ANSWER TO QUESTION 1 IS 'YES'

Let $(X, G)$ be a space of orderings. We say the pp formula $\exists \underline{t} P(\underline{t})$, $\underline{t} = (t_1, \dots, t_n)$, is *product free and 1-related* if:

(1) The atomic formulas appearing in $P(\underline{t})$ which actually involve variables are expressible in the form '$t_i \in D\langle a, b\rangle$' or '$1 \in D\langle at_i, bt_j\rangle$', $i \neq j$, or '$t_i \in D\langle at_j, bt_k\rangle$',

$i, j, k$ distinct, where $a, b$ are elements of $G$. For each $i$ we allow any (finite) number of atoms of the form $t_i \in D\langle a, b\rangle$, but require that for each $i, j$ with $i \neq j$ there is at most one atom involving both $t_i$ and $t_j$.

Requirement (1) allows us to construct from $P(\underline{t})$ a graph with vertices $1, \dots, n$ with an edge joining $i$ and $j$ iff there is some (necessarily unique) atom of $P(\underline{t})$ involving both $t_i$ and $t_j$. The second requirement is:

(2) The graph associated to $P(\underline{t})$ contains no cycles other than the 3-cycles arising from atoms of the form '$t_i \in D\langle at_j, bt_k\rangle$'.

**2.1 Theorem.** *Suppose $(X, G)$ is a space of orderings of finite stability index. Then for any product free 1-related pp formula $\exists \underline{t} P(\underline{t})$ with parameters in $G$ the answer to Question 1 is 'yes'.*

To see how this fits with what was previously known, we note that the statement '$b \in D(\phi)$', for $\phi = \langle a_1, \dots, a_n\rangle$, $n \geq 3$ is equivalent to the statement

$$\exists t_1, \dots, \exists t_{n-2} \text{ such that } b \in D\langle a_1, t_1\rangle \wedge t_1 \in D\langle a_2, t_2\rangle \wedge \cdots \wedge t_{n-2} \in D\langle a_{n-1}, a_n\rangle,$$

a product free 1-related pp formula whose associated graph is linear with $n-2$ vertices. The statement '$\langle a_1, \dots, a_n\rangle$ is isotropic', since it is equivalent to the statement '$-a_1 \in D\langle a_2, \dots, a_n\rangle$', can be expressed as a product free 1-related pp formula whose associated graph is linear with $n-3$ vertices. Similarly, the statement '$\cap_{i=1}^m D(\phi_i) \neq \emptyset$' can be expressed as a product free 1-related pp formula whose associated graph is 'star-shaped' ($m'$ linear graphs emanating from a common vertex, where $m'$ is the number of indices $i$ with $\dim(\phi_i) \geq 3$). Clearly there are a great many product free 1-related pp formulas which are not of these two types. Also, whereas for a general product free 1-related pp formula there is no restriction on the number of atoms of the form $t_i \in D\langle a, b\rangle$ appearing, in the pp formulas corresponding to '$\phi$ is isotropic' and '$\cap_{i=1}^m D(\phi_i) \neq \emptyset$', only a few such atoms appear, and only at particular locations on the graph.

Thus, except for the unpleasant requirement that $(X, G)$ has finite stability index, Theorem 2.1 extends [16, Th. 1.4] and [17, Th. 2.1] considerably. Although the proof requires this assumption, it may in fact not be necessary. In any case, the most interesting examples of spaces of orderings, e.g., the space of orderings of a formally real function field over a real closed field, satisfy this requirement automatically, e.g., see [18, Th. 3.6.3].

Concerning the proof of Theorem 2.1, it is a modification of the proof of the abovementioned result in [17] (which, in turn, is a modification of the proof of the Isotropy Theorem in [16]). We also use a result of Bröcker, namely [6, Prop. 5.24]. This is where the assumption on the stability index is used. We remark that the proof of [6, Prop. 5.24] given in [6] uses the Isotropy Theorem.

*Proof of Theorem 2.1.* Let the formula be $\exists \underline{t} P(\underline{t})$, $\underline{t} = (t_1, \dots, t_n)$. Let $s$ be the stability index of $(X, G)$. If $s \leq 1$ the result is true by [20, Prop. 2.3]. Thus we can assume $s \geq 2$. Assume the formula $\exists \underline{t} P(\underline{t})$ does not hold in $(X, G)$. According to

[20, Prop. 2.2], we may assume that $(X,G)$ is minimal with the property, i.e., that $\exists \underline{t} P(\underline{t})$ holds in any proper subspace $(Y,H)$ of $(X,G)$. If $(X,G)$ has finite chain length then we are done. (By the Structure Theorem, a space of orderings of finite chain length and finite stability index is necessarily finite.) Otherwise, for any large $k$, we have elements $a_0,\dots,a_k$ in $G$, $a_0=1$, $a_k=-1$ such that $D\langle 1,a_{j-1}\rangle \subsetneqq D\langle 1,a_j\rangle$ for $j=1,\dots,k$. Then, for each $j$, $a_{j-1}a_j\neq 1$ so we have elements $t_{ij}$ in $G$, $i=1,\dots,n$ satisfying the formula $P(\underline{t})$ in the subspace $U(a_{j-1}a_j)$ of $(X,G)$. Thus, for each atom of the form '$t_i \in D\langle c,d\rangle$', we have

$$\langle c,d\rangle \otimes \langle 1,a_{j-1}a_j\rangle \cong \langle t_{ij}, cdt_{ij}\rangle \otimes \langle 1,a_{j-1}a_j\rangle, \tag{1}$$

for each atom of the form '$1 \in D\langle ct_i, dt_{i'}\rangle$' with $i\neq i'$, we have

$$\langle ct_{ij}, dt_{i'j}\rangle \otimes \langle 1,a_{j-1}a_j\rangle \cong \langle 1, cdt_{ij}t_{i'j}\rangle \otimes \langle 1,a_{j-1}a_j\rangle, \tag{2}$$

and for each atom of the form '$t_i \in D\langle ct_{i'}, dt_{i''}\rangle$' with $i,i',i''$ distinct, we have

$$\langle ct_{i'j}, dt_{i''j}\rangle \otimes \langle 1,a_{j-1}a_j\rangle \cong \langle t_{ij}, cdt_{ij}t_{i'j}t_{i''j}\rangle \otimes \langle 1,a_{j-1}a_j\rangle. \tag{3}$$

As in [16, proof of Th. 1.4],

$$\langle a_0a_1,\dots,a_{k-1}a_k\rangle \cong \langle 1,\dots,1,-1\rangle. \tag{4}$$

We are interested in the forms $\tau_i := \langle t_{i1},\dots,t_{ik}\rangle$, $i=1,\dots,n$. It is important to observe that the form $\oplus_{j=1}^k t_{ij}\langle a_{j-1}a_j,-1\rangle$ has signature 2 or $-2$ at each ordering. This follows from the fact that at each ordering, exactly $k-1$ of the $a_{j-1}a_j$, $j=1,\dots,k$ are positive.

Consider first the case of an atom '$t_i \in D\langle c,d\rangle$'. Adding the isometries (1), $j=1,\dots,k$, using (4) and

$$\oplus_{j=1}^k t_{ij}\langle 1,a_{j-1}a_j\rangle \sim \oplus_{j=1}^k t_{ij}\langle 1,1\rangle \oplus \oplus_{j=1}^k t_{ij}\langle a_{j-1}a_j,-1\rangle, \tag{5}$$

we obtain

$$(2k-2)\times\langle c,d\rangle \sim 2\langle 1,cd\rangle\otimes\tau_i \oplus \langle 1,cd\rangle \otimes (\oplus_{j=1}^k t_{ij}\langle a_{j-1}a_j,-1\rangle).$$

It follows that, on the subspace $U(cd)$, $\oplus_{j=1}^k t_{ij}\langle a_{j-1}a_j,-1\rangle$ is divisible by 2, so has the form $2\times\langle e\rangle$, for some $e\in G$, and consequently that:

$$(k-1)\times\langle c,d\rangle \sim \langle 1,cd\rangle\otimes\tau_i \oplus e\langle 1,cd\rangle. \tag{6}$$

The case of an atom of the form '$1 \in D\langle ct_i, dt_{i'}\rangle$' is more complicated. By [6, Prop. 5.24], we have a form $\psi_i$ of dimension $4^{s-1}$ with $2^{s-2} \times \oplus_{j=1}^{k} t_{ij} \langle a_{j-1}a_j, -1\rangle \cong \psi_i$. Thus, adding the isometries (2), $j = 1, \dots, k$, multiplying by $2^{s-2}$, and using (4) and (5), we obtain

$$(7)\quad 2^{s-1} \times (c\tau_i \oplus d\tau_{i'}) \oplus c\psi_i \oplus d\psi_{i'} \sim 2^{s-1}(k-1) \times \langle 1\rangle \oplus 2^{s-2} \times cd \oplus_{j=1}^{k} t_{ij}t_{i'j} \langle 1, a_{j-1}a_j\rangle.$$

Similarly, in the case of an atom of the form '$t_i \in D\langle ct_{i'}, dt_{i''}\rangle$', we obtain

$$(8)\quad 2^{s-1} \times (c\tau_{i'} \oplus d\tau_{i''}) \oplus \psi_{i'} \oplus \psi_{i''} \sim 2^{s-1} \times \tau_i \oplus \psi_i \oplus 2^{s-2} \times cd \oplus_{j=1}^{k} t_{ij}t_{i'j}t_{i''j} \langle 1, a_{j-1}a_j\rangle.$$

Computing dimensions in (6), (7) and (8), this establishes the following three basic facts:

**Fact 1:** for any atom of the form '$t_i \in D\langle c, d\rangle$' appearing in $P(\underline{t})$, the form $(k-1) \times \langle c, d\rangle - \tau_i$ has Witt index at least $k-2$ if $k \geq 2$.

**Fact 2:** For any atom of the form '$1 \in D\langle ct_i, dt_{i'}\rangle$' appearing in $P(\underline{t})$, the form $c\tau_i \oplus d\tau_{i'} - (k-1) \times \langle 1\rangle$ has Witt index at least $k - 2^{s-1}$ if $k \geq 2^{s-1}$.

**Fact 3:** For any atom of the form '$t_i \in D\langle ct_{i'}, dt_{i''}\rangle$' appearing in $P(\underline{t})$, the form $c\tau_{i'} \oplus d\tau_{i''} - \tau_i$ has Witt index at least $k - 2^{s-1} - 2^{s-2}$ if $k \geq 2^{s-1} + 2^{s-2}$.

The contradiction is obtained by using these facts, with $k$ sufficiently large, to prove that the formula $\exists \underline{t} P(\underline{t})$ holds in $(X, G)$. Clearly we can reduce to the case where the graph of $P(\underline{t})$ is connected. (Otherwise, we work separately with the formulas corresponding to the connected components of the graph.)

We fix a vertex, say the vertex 1 corresponding to the variable $t_1$. Relabeling, we can assume the immediate neighbors of this vertex (if any) are the vertices $2, \dots, \ell$ corresponding to the variables $t_2, \dots, t_\ell$. Removing the vertex 1 and the edges joining 1 and $i$, $i = 2, \dots, \ell$, as well as the edges joining $i$ and $i'$ in the cases where $1, i, i'$ form a 3-cycle, creates $\ell - 1$ connected subgraphs, each with fewer vertices. Denote by $P(\underline{t}^{(i)})$ the conjunction of the atoms in $P(\underline{t})$ which involve only variables $t_j$ with $j$ in the $i$-th subgraph, $i = 2, \dots, \ell$.

We choose $k > v$, $v = 2n_1 + 2^{s-1}n_2 + (2^{s-1} + 2^{s-2})n_3$ where $n_1$ is the number of atoms of the first type, $n_2$ is the number of atoms of the second type and $n_3$ is the number of atoms of the third type. We prove, by induction on $n$, the existence of a subform of $\tau_1'$ of $\tau_1$ of dimension $k - v$ with the property that for any $t_1 \in D(\tau_1')$ there exists $t_2, \dots, t_n \in G$ such that $P(\underline{t})$ holds in $(X, G)$. By induction on $n$, for $i = 2, \dots, \ell$ we have a subform $\tau_i'$ of $\tau_i$ of dimension $\geq k - v_i$, with the property that for each $t_i \in D(\tau_i')$ there exists $t_j \in G$ for each $j \neq i$ in the subgraph corresponding to $i$ such that the formula $P(\underline{t}^{(i)})$ holds for this choice of $t$'s.

By Fact 1, for each of the finite number of atoms of the form '$t_1 \in D\langle c_j, d_j\rangle$', we have a decomposition of $\tau_1$ of the form $\tau_1 \cong \alpha_j \oplus \alpha_j'$ where $\alpha_j$ is a subform of

$(k-1)\times\langle c_j, d_j\rangle$ of dimension $k-2$. By Fact 2, for each atom of the form '$1 \in D\langle a_i t_1, b_i t_i\rangle$', $i \in \{2,\dots,\ell\}$, $a_i\tau_1 \oplus b_i\tau_i' - (k-1)\times\langle 1\rangle$ has Witt index $\geq k - 2^{s-2} - v_i$, so we have a decomposition $\tau_1 \cong \beta_i \oplus \beta_i'$, $\dim(\beta_i) = k - 2^{s-1} - v_i$, with $-a_i\beta_i$ a subform of $b_i\tau_i' - (k-1)\times\langle 1\rangle$. By Fact 3, for each atom of the form '$t_1 \in D\langle a_i t_i, a_{i'} t_{i'}\rangle$', $a_i\tau_i' \oplus a_{i'}\tau_{i'}' - \tau_1$ has Witt index $\geq k - 2^{s-2} - 2^{s-2} - v_i - v_{i'}$, so we have a decomposition $\tau_1 \cong \gamma_{ii'} \oplus \gamma_{ii'}'$, $\dim(\gamma_{ii'}) = k - 2^{s-1} - 2^{s-2} - v_i - v_{i'}$, with $\gamma_i$ a subform of $a_i\tau_i' \oplus a_{i'}\tau_{i'}'$. By [17, Lemma 2.4], the forms $\alpha_j$, $\beta_i$, $\gamma_{ii'}$ have a common subform $\tau_1'$ of dimension $k - (2r_1 + 2^{s-1}r_2 + (2^{s-1}+2^{s-2})r_3 + \sum_{i=2}^{\ell} v_i)$ where $r_1$ is the number of atoms '$t_1 \in D\langle ac_j, d_j\rangle$', $r_2$ is the number of atoms '$1 \in D\langle a_i t_1, b_i t_i\rangle$', and $r_3$ is the number of atoms '$t_1 \in D\langle a_i t_i, a_{i'} t_{i'}\rangle$'. Clearly $2r_1 + 2^{s-1}r_2 + (2^{s-1}+2^{s-2})r_3 + \sum_{i=2}^{\ell} v_i = v$. Finally, if $t_1 \in D(\tau_1')$, then $t_1 \in D\langle c_j, d_j\rangle$ for each $j$ and, for each $i \in \{2,\dots,\ell\}$ not part of a 3-cycle involving 1, $1 \in D\langle a_i t_1, b_i t_i\rangle$ for some $t_i \in D(\tau_i')$. At the same time, for each $i, i'$ part of a 3-cycle involving 1, $t_i \in D\langle a_i t_i, a_{i'} t_{i'}\rangle$ for some $t_i \in D(\tau_i')$ and some $t_{i'} \in D(\tau_{i'}')$. The result follows now, using the inductive hypothesis. □

## 3. Products of binary value sets

Let $(X,G)$ be a space of orderings. Theorem 2.1 settles Question 1 for pp formulas $\exists \underline{t} P(\underline{t})$ which are product free and 1-related. Question 1 is also settled for pp formulas involving just one quantified variable [20, Cor. 2.5]. The simplest sorts of pp formula for which Question 1 is open are those involving just two quantified variables $\underline{t} = (t_1, t_2)$ which are either of the type

$$\exists t_1 \exists t_2\ t_1 \in D\langle a,b\rangle \ \wedge\ t_2 \in D\langle c,d\rangle \ \wedge\ t_1t_2 \in D\langle e,f\rangle \tag{1}$$

or of the type

$$\exists t_1 \exists t_2\ t_1 \in D\langle a,bt_2\rangle \ \wedge\ t_2 \in D\langle c,dt_1\rangle \ \wedge\ t_1 \in D\langle e,f\rangle, \tag{2}$$

$a,b,c,d,e,f \in G$.

**3.1 Proposition.** *Each of the pp formulas (1) and (2) above is equivalent to a formula of the type*

$$b \in D\langle 1,a_1\rangle D\langle 1,a_2\rangle D\langle 1,a_3\rangle$$

*for suitable* $a_1, a_2, a_3, b \in G$.

*Proof.* (1) is obviously equivalent to the statement $1 \in D\langle a,b\rangle D\langle c,d\rangle D\langle e,f\rangle$ which, in turn, is equivalent to the statement $ace \in D\langle 1,ab\rangle 1,cd\rangle D\langle 1,ef\rangle$.

If (2) holds in $(X,G)$ then there exists $t_1 \in G$ such that $t_1 \in D\langle a,bc,bdt_1\rangle$, i.e., the form $\langle a,bc,-t_1,bdt_1\rangle$ is isotropic, i.e., $t_1 D\langle 1,-bd\rangle \cap D\langle a,bc\rangle \neq \emptyset$. Since we assume at the same time that $t_1 \in D\langle e,f\rangle$, this yields $D\langle e,f\rangle D\langle 1,-bd\rangle \cap D\langle a,bc\rangle \neq \emptyset$, i.e., $1 \in D\langle a,bc\rangle D\langle e,f\rangle D\langle 1,-bd\rangle$, i.e.,

$$ae \in D\langle 1,abc\rangle D\langle 1,ef\rangle D\langle 1,-bd\rangle. \tag{3}$$

Conversely, if (3) holds, one checks that each step in the above argument are reversible, i.e., there exist $t_1, t_2 \in G$ satisfying the conditions in (2). $\square$

Thus there is a natural interest in formulas of the type $b \in \prod_{i-1}^{n} D\langle 1, a_i\rangle$. Refer to Question 2 in the preface. We begin the study of such formulas by comparing the product of the binary value sets $\prod_{i=1}^{n} D\langle 1, a_i\rangle$ with the value set $D(\phi)$ where $\phi$ denotes the $n$-fold Pfister form $\phi = \otimes_{i=1}^{n}\langle 1, a_i\rangle$. Clearly $\prod_{i=1}^{n} D\langle 1, a_i\rangle$ is a subgroup of $D(\phi)$ with equality holding if $n = 1$.

The statement '$d \in D(\phi)$' can be expressed as a pp formula and the local-global property (Question 1) holds for this particular pp formula [16, Th. 1.4]. Thus the validity of the statement '$b \in D(\phi)$' can be determined by checking it locally on each finite subspace of $(X, G)$. In fact, since

$$d \in D(\phi) \text{ iff } \forall \sigma \in X \ \sigma a_i = 1,\ i = 1, \ldots, n \ \Rightarrow \ \sigma d = 1,$$

it suffices to check the statement on each singleton subspace of $(X, G)$.

The statement '$d \in \prod_{i=1}^{n} D\langle 1, a_i\rangle$' can also be expressed as a pp formula, namely, it is equivalent to the statement '$\exists t_1, \ldots, t_{n-1} \in G$ such that $t_i \in D\langle 1, a_i\rangle$, $i = 1, \ldots, n-1$, and $dt_1 \ldots t_{n-1} \in D\langle 1, a_n\rangle$'. This statement is simple to check when $n = 2$:

**3.2 Proposition.** *For $n = 2$, the answer to Question 2 is 'yes'.*

*Proof.* When $n = 2$ the statement in question has only one quantified variable, so the result follows from [20, Cor. 2.5]. Alternatively, the statement is equivalent to the assertion that $d \in D\langle 1, a_1, -da_2\rangle$ so one can apply the Isotropy Theorem [16, Th. 1.4] directly to obtain the desired conclusion. $\square$

We denote the stability index of $(X, G)$ by $s$.

**3.3 Proposition.** *If $s \leq 1$ then $\prod_{i=1}^{n} D\langle 1, a_i\rangle = D(\phi)$.*

*Proof.* Let $d \in D(\phi)$. Thus $\cap_{i=1}^{n} U(a_i) \subseteq U(d)$. Since $s \leq 1$, every continuous function from $X$ to $\{1, -1\}$ belongs to $G$. Thus we can choose $t_1, \ldots, t_n \in G$ such that $t_1 = d$ on $U(-a_1)$, $t_1 = 1$ on $U(a_1)$, $t_2 = d$ on $U(a_1) \cap U(-a_2)$, $t_2 = 1$ on $U(-a_1) \cup U(a_2)$, $t_3 = d$ on $U(a_1) \cap U(a_2) \cap U(-a_3)$, $t_3 = 1$ on $U(-a_1) \cup U(-a_2) \cup U(a_3)$, etc.. Then it is clear that $t_i \in D\langle 1, a_i\rangle$, $i = 1, \ldots, n$ and $d = t_1 \ldots t_n$. $\square$

Suppose $(X, G)$ has stability index $s \geq 2$. Thus there exist $a, b \in G$ with $1 \notin D\langle a, b, ab\rangle$ [18, Th. 3.4.2]. We claim the conclusion of Proposition 3.3 fails with $n = 2$, $a_1 = a$, $a_2 = -a$. To prove this it suffices to show that $ab \notin D\langle 1, a\rangle D\langle 1, -a\rangle$. Suppose this is not the case, so $ab = t_1 t_2$ with $t_1 \in D\langle 1, a\rangle$, $t_2 \in D\langle 1, -a\rangle$. Then $abt_1 = t_2$ is positive whenever $a$ is negative, so $bt_1$ is negative whenever $a$ is negative. It follows that $a \in D\langle 1, bt_1\rangle$, so $ab \in D\langle b, t_1\rangle \subseteq D\langle 1, a, b\rangle$, i.e., $1 \in D\langle a, b, ab\rangle$, a contradiction.

The fact that $\prod_{i=1}^{n} D\langle 1, a_i\rangle = D(\phi)$ can fail so easily even when $n = 2$ suggests that in looking for counterexamples perhaps we should look instead at some larger subgroup of $D(\phi)$. The subgroup we consider is the product $\prod_{\delta \in \{0,1\}^n} D\langle 1, a^{\delta}\rangle$, where $a^{\delta} := a_1^{\delta_1} \ldots a_n^{\delta_n}$.

**3.4 Proposition.** *If $(X,G)$ is a fan then $\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle = D(\phi)$.*

*Proof.* If $a^\delta = -1$ for some $\delta\in\{0,1\}^n$ then $\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle = D(\phi) = G$. Otherwise, $D\langle 1,a^\delta\rangle = \{1,a^\delta\}$ for all $\delta$ and $\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle = D(\phi) = \{a^\delta \mid \delta \in \{0,1\}^n\}$. $\square$

The conclusion of Proposition 3.4 fails when $s=2$, $n\geq 2$ for group extensions which are not fans:

**3.5 Proposition.** *Suppose $(X,G)$ is a group extension by a cyclic group of order 2 of a space of orderings $(\overline{X},\overline{G})$ of stability index 1 which is not a fan. Then:*

1. *If $a_1,\dots,a_n\in\overline{G}$ and $a^\delta\neq -1$ for all $\delta\in\{0,1\}^n$ and $\phi$ is isotropic, then $\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle = \overline{G}$, $D(\phi)=G$.*
2. *In all other cases (i.e., if either $a_i\notin\overline{G}$ for some $i$, or if $a_1,\dots,a_n\in\overline{G}$ and $\phi$ is anisotropic, or if $a_1,\dots,a_n\in\overline{G}$ and $a^\delta=-1$ for some $\delta$), $\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle = D(\phi)$.*

*Proof.* Clear from the behavior of forms under group extension; i.e., see [18, proof of Th. 4.1.1(2)]. $\square$

For finite spaces of orderings of stability index 2, the obstruction described in part (1) of Proposition 3.5 is the only one in the following sense:

**3.6 Corollary.** *Suppose $(X,G)$ is a finite space of orderings of stability index $s=2$ and $d\in D(\phi)$. Then the following are equivalent:*

1. *$d\in\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle$.*
2. *For each connected component $(Y,H)$ of $(X,G)$ which is not a fan, if the images of the $a_i$ in $H$ belong to $\overline{H}$, where $(\overline{Y},\overline{H})$ denotes the residue space of $(Y,H)$, and the image of each $a^\delta$ in $H$ is not equal to $-1$ and if the form $\phi$ is isotropic over $(Y,H)$, then the image of $d$ in $H$ belongs to $\overline{H}$.*

The Structure Theorem implies that any finite space of orderings is the direct sum of its connected components. The proof of Corollary 3.6 uses this fact. See [16] for the definition of connected components and residue spaces and for the relationship between these objects and group extensions.

One naturally wonders if the conclusion of Corollary 3.6 remains true when $(X,G)$ is an arbitrary space of orderings of stability index 2. This will be the case iff the validity of the pp formula $d\in\prod_{\delta\in\{0,1\}^n} D\langle 1,a^\delta\rangle$ depends only on its local validity on each finite subspace.

## 4. THE CASE $F = R(x,y)$

Let $(X,G)$ denote the space of orderings of the field $F=R(x,y)$, the rational function field in two variables over $R$, where $R$ is a real closed field. The stability index of $(X,G)$ is 2 [18, Th. 3.6.3].

We use the same symbol to denote a non-zero element of $F$ and its image in $G = F^{\cdot}/\Sigma F^{2^{\cdot}}$. Each $f \in G$ is represented by a nonzero element $f \in R[x,y]$ which is square free. We restrict our attention to the case where $n = 2$, $a_1 = x$, $a_2 = y$. Thus

$$\begin{cases} \phi = \langle 1, x\rangle \otimes \langle 1, y\rangle \\ \prod_{\delta\in\{0,1\}^n} D\langle 1, a^{\delta}\rangle = D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle \end{cases}.$$

**4.1 Theorem.** *Assume that $f \in R[x,y]$ is square free. Then:*

*(1) $f \in D\langle 1,x\rangle \otimes \langle 1,y\rangle$ iff $f$ is non-negative on the first quadrant of $R^2$. In this case, no irreducible factor of $f$ changes sign on the first quadrant. Conversely, if no irreducible factor of $f$ changes sign on the first quadrant, then either $f$ belongs to $D\langle 1,x\rangle \otimes \langle 1,y\rangle$ or $-f$ belongs to $D\langle 1,x\rangle \otimes \langle 1,y\rangle$.*

*(2) $f \in D\langle 1,x\rangle D\langle 1,y\rangle D\langle 1,xy\rangle$ iff $f$ is non-negative on the first quadrant and no irreducible factor of $f$ changes sign on each of the other three quadrants.*

*Proof.*

(1) If $f \geq 0$ on the first quadrant of $R^2$ then, by the Tarski Transfer Principle, $\sigma x = 1$ and $\sigma y = 1 \Rightarrow \sigma f = 1$ for all $\sigma \in X$, so $f \in D\langle 1,x\rangle \otimes \langle 1,y\rangle$. Suppose now that $f(a,b) < 0$ for some real $a, b > 0$. A standard construction [3, Prop. 7.6.2] produces an ordering $\sigma$ of $R(x,y)$ which specializes to $(a,b)$. Then $\sigma x = 1$, $\sigma y = 1$, $\sigma f = -1$, which proves that $f \notin D\langle 1,x\rangle \otimes \langle 1,y\rangle$. If some irreducible factor $p$ of $f$ changes sign in the first quadrant, then, by continuity (using the fact that the first quadrant is semialgebraically connected), $p$ vanishes at infinitely many points in the first quadrant. In particular, there is a non-singular point on the curve $p = 0$ in the first quadrant where none of the other irreducible factors of $f$ vanish. At this point, $p$ changes sign but the other irreducible factors of $f$ do not change sign, so $f$ changes sign at this point. Conversely, if none of the irreducible factors change sign on the first quadrant, then $f$ does not change sign on the first quadrant, so $f$ or $-f$ belongs to $D\langle 1,x\rangle \otimes \langle 1,y\rangle$.

(2) By (1) we can arrange things (replacing $p$ by $-p$ is necessary) so that each irreducible factor of $f$ is non-negative on the first quadrant. Claim: If $p \in R[x,y]$ is irreducible and non-negative on the first quadrant and also does not change sign on some other quadrant, then $p \in D\langle 1,x\rangle D\langle 1,y\rangle D\langle 1,xy\rangle$. Suppose, for example, that $p$ is $\geq 0$ on the first quadrant and does not change sign on the second quadrant. If $p$ is $\geq 0$ on the second quadrant, this implies $p \geq 0$ whenever $y > 0$, so $p \in D\langle 1,y\rangle$. On the other hand, if $p$ is $\leq 0$ on the second quadrant, then the $xp \geq 0$ whenever $y > 0$, so $p \in xD\langle 1,y\rangle$. The remaining cases are similar. This proves the claim. By the claim, if no irreducible factor of $f$ changes sign on each of the other quadrants, then each irreducible factor of $f$ belongs to $D\langle 1,x\rangle D\langle 1,y\rangle D\langle 1,xy\rangle$, and consequently, $f \in D\langle 1,x\rangle D\langle 1,y\rangle D\langle 1,xy\rangle$. Finally, suppose $f$ has an irreducible factor $p$ which changes sign on each of the other quadrants. Suppose $f \in D\langle 1,x\rangle D\langle 1,y\rangle D\langle 1,x\rangle$. Clearing denominators yields an equation of the form $g^2 f = (\alpha_0 + \alpha_1 x)(\beta_0 + \beta_1 y)(\gamma_0 + \gamma_1 xy)$ for some nonzero $g \in R[x,y]$ and $\alpha_i, \beta_i, \gamma_i \in \sum R[x,y]^2$, $i = 0, 1$. Choose such an equation with $g$ having minimal degree. If $p$ divides $\alpha_0 + \alpha_1 x$ then, using the fact

that $p$ has infinitely many zeroes in the 4th quadrant, we see that $p$ divides $\alpha_i$ and consequently that $p^2$ divides $\alpha_i$, for $i = 0, 1$. Then $p$ also divides $g$, and dividing both sides of our equation by $p^2$ we have a contradiction to the minimal choice of the degree of $g$. Thus $p$ cannot divide $\alpha_0 + \alpha_1 x$. A similar argument shows that $p$ cannot divide $\beta_0 + \beta_1 y$ or $\gamma_0 + \gamma_1 xy$. This is a contradiction. $\square$

**4.2 Example.** Let $f_1 = 1 + 2x + 2y + x^2 + y^2$, $f_2 = 2x + 2y + x^2 + y^2$, $f_3 = 1 - xy + x^2y + xy^2$. Then

$$\begin{cases} f_1 \in D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle, \\ f_2 \in D\langle 1, x\rangle \otimes \langle 1, y\rangle, \ \ f_2 \notin D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle, \\ f_3 \in D\langle 1, x\rangle \otimes \langle 1, y\rangle, \ \ f_3 \notin D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle. \end{cases}$$

**4.3 Corollary.** *If $f \in D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle$ fails to hold, then it already fails to hold in some finite subspace of $(X, G)$.*

*Proof.* We can assume that $f$ is in $R[x, y]$ and that $f$ is square free. If $f \notin D\langle 1, x\rangle \otimes \langle 1, y\rangle$ then $f \in D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle$ fails already in some singleton subspace of $(X, G)$. Thus we can assume that $f \in D\langle 1, x\rangle \otimes \langle 1, y\rangle$. By Theorem 4.1 there is some irreducible factor $p$ of $f$ which is non-negative on the first quadrant, but changes sign on each of the other quadrants. Consider the valuation $v$ of $F = R(x, y)$ with valuation ring equal to $R[x, y]_{(p)}$, the localization of $R[x, y]$ at the prime ideal generated by $p$. Denote by $(X_v, G_v)$ the subspace of $(X, G)$ consisting of orderings compatible with $v$ [18, Th. 3.6.1]. One checks easily that the hypothesis of Proposition 3.6 (1) holds in the subspace $(X_v, G_v)$. ($v(x) = v(y) = 0$, $v(p) = 1$, $v(q) = 0$ for other irreducible factors of $f$, if any. Thus when we view $x, y, f$ as elements of $G_v$, we see that $x, y \in \overline{G}_v$, $f \notin \overline{G}_v$. There is an element of $X_v$ making $x$ positive, and similarly for $y$, $xy$. At the same time, there is no element of $X_v$ making $x$ and $y$ both positive, so $\langle 1, x\rangle \otimes \langle 1, y\rangle$ is isotropic in $(X_v, G_v)$.) Thus, by Proposition 3.5 (1), $f \in D\langle 1, x\rangle D\langle 1, y\rangle D\langle 1, xy\rangle$ fails to hold already in the subspace $(X_v, G_v)$. To obtain a finite subspace with this property, fix orderings $\alpha, \beta, \gamma \in \overline{X}_v$ with $\alpha x = 1$, $\beta y = 1$, and $\gamma(xy) = 1$ and take $(Y, H)$ to be the pull-back in $(X_v, G_v)$ of the subspace $(\overline{Y}, \overline{H})$ of $(\overline{X}_v, \overline{G}_v)$ where $\overline{Y} = \{\alpha, \beta, \gamma\}$. $\square$

We remark that Theorem 4.1 and Corollary 4.3 remain true with $x, y$ replaced by any non-zero $t, u \in R[x, y]$ having the property that the four ‘quadrants’ in $R^2$ defined by $t, u$ are semialgebraically connected. The proof is exactly the same.

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