POSITIVITY IN POWER SERIES RINGS AND APPLICATIONS TO EQUIVARIANT SUMS OF SQUARES

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Abstract. We extend and generalize results of Scheiderer (2006) on the representation of polynomials nonnegative on two-dimensional basic closed semialgebraic sets. Our extension covers some situations where the defining polynomials do not satisfy the transversality condition. Such situations arise naturally when we consider semialgebraic sets invariant under finite group actions.

1. Introduction

Let $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \cdots, x_n]$ be the ring of polynomials in n variables and real coefficients. Write $\sum \mathbb{R}[\mathbf{x}]^2$ for the set of all sums of squares of polynomials from $\mathbb{R}[\mathbf{x}]$. Note that $\sum \mathbb{R}[\mathbf{x}]^2$ is a subsemiring of $\mathbb{R}[\mathbf{x}]$. Any subsemiring of $\mathbb{R}[\mathbf{x}]$ containing $\sum \mathbb{R}[\mathbf{x}]^2$ is called a **preordering**. In other words, preorderings are the subsets of $\mathbb{R}[\mathbf{x}]$ that contain $\sum \mathbb{R}[\mathbf{x}]^2$ and are closed for addition and multiplication.

For a given subset $S = \{f_1, ..., f_s\}$ of $\mathbb{R}[\mathbf{x}]$ write T_S for the smallest preordering containing S and K_S for the subset of all $x \in \mathbb{R}^n$ satisfying $f_1(x) \geq 0, \ldots, f_s(x) \geq 0$. We say that T_S is the **preordering generated by** S and K_S is the **basic closed semialgebraic set generated by** S; note that the set K_S is uniquely determined by the set T_S but T_S is not uniquely determined by K_S.

Write **Psd**(K) for the set of all elements of $\mathbb{R}[\mathbf{x}]$ that are nonnegative on the set K. We always have that $T_S \subseteq \mathrm{Psd}(K_S)$. The preordering T_S is **saturated** if $T_S = \mathrm{Psd}(K_S)$. In this paper, we investigate what geometric properties of S imply that T_S is saturated. This line of investigation has been pursued by Scheiderer in a series of papers. In [10] Scheiderer showed:

Theorem 1. *If dim*$(K_S) \geq 3$*, then there exists a polynomial* $p \in \mathbb{R}[\mathbf{x}]$ *such that* $p \geq 0$ *on* $\mathbb{R}^n$ *but* $p \notin T_S$ *(so* T_S *cannot be saturated).*

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It follows from this result that the above question should be studied only under the assumption that $\dim(K_S) \leq 2$. The case when $\dim(K_S) \leq 1$ has been extensively studied elsewhere (see e.g. [4], [5], [11]). Let us focus on the case $\dim(K_S) = 2$. In this paper, we study the affine 2-dimensional case, i.e. the case when $n = 2$ and K_S has non-empty interior. In [10] Scheiderer further showed:

Theorem 2. *If $n = 2$ and K_S contains a cone of dimension 2, then there exists a polynomial $p \in \mathbb{R}[x, y]$ such that $p \geq 0$ on $\mathbb{R}^2$ but $p \notin T_S$ (so T_S cannot be saturated).*

This result motivates studying the above question under the further assumption that $K_S \subseteq \mathbb{R}^2$ does not contain a cone of dimension 2. In [4] we asked specifically, what if K_S is compact, e.g. the unit square $S = \{1 + x, 1 - x, 1 + y, 1 - y\}$ ([4]; open problem 7), or K_S is an unbounded cylinder with compact base ([4]; open problem 6)? While the second problem remains completely open, the first problem has been partially solved by the following result of Scheiderer [12, Corollary 3.3]:

Theorem 3. *Let $S = \{g_1, ..., g_s\}$ be irreducible polynomials in $\mathbb{R}[x, y]$, let C_i be the plane affine curve $g_i = 0$ $(i = 1, ..., s)$. Assume:*

1. *K_S is compact*
2. *C_i has no real singular points $(i = 1, ..., s)$*
3. *the real points of intersection of any two of the C_i are transversal, and no three of the C_i intersect in a real point.*

Then T_S is saturated.

The main purpose of this paper is to explain how saturation can hold in other cases as well, e.g., if the compact set is defined by $S = \{x, 1 - x, y, x^2 - y\}$ or by $S = \{1 + x, 1 - x, y, x^2 - y\}$. In these examples, the boundary curves $y = 0$ and $y = x^2$ share a common tangent at the origin. These two cases are instances of assertions (3) and (4) of Corollary 8 which is our main result. It is a generalization of Theorem 3.

The motivation for this generalization comes from examples which arise naturally while studying semialgebraic sets $K_{S'}$ described by a set S' of polynomials invariant under the action of a finite group G. The corresponding preordering $T_{S'}$ will typically not be saturated but it can still be “saturated for invariant polynomials” (we refer to this as G-saturation, see Section 3). The orbit map π (see [3]) relates the G-saturation of $T_{S'}$ to the saturation of certain preordering $T_{\tilde{S}'}$ corresponding to $\pi(K_{S'}) = K_{\tilde{S}'}$. However, orbit maps will in general not preserve transversal intersections of curves. For example, the unit square is also described by the D_4-invariant set $S' = \{2-x^2-y^2, (1-x^2)(1-y^2)\}$. The image of the unit square in the orbit space is a planar affine semialgebraic set with "zero-angle" intersections. Our Corollary establishes that the corresponding preordering is saturated. This in turn establishes the D_4- saturation property for $T_{S'}$ (see Section 3 for details).

Corollary 8 does not cover all interesting cases: in the Concluding Remarks, we discuss systematically various two-dimensional affine cases; listing cases that remain open. We provide various naturally occuring affine examples for which our Corollary is inconclusive.

Our main Corollary does not cover the cases when some $g_i \in S$ is reducible or has singular zeros or when K_S is noncompact or nonaffine (a nonaffine case occurs e.g. when studying the $\mathbb{Z}_4$-saturation property for the $T_{S'}$ above). These cases will be considered in future work.

2. Saturation in dimension two

Here, our results focus on the case of a *compact* basic closed semialgebraic set. In [11, Cor. 3.17], Scheiderer proves a useful 'local-global' criterion for deciding when a polynomial non-negative on a compact basic closed semialgebraic set lies in the associated preordering of the polynomial ring:

Theorem 4. *Suppose $f, g_1, \dots, g_s \in \mathbb{R}[\mathbf{x}]$, the subset K of $\mathbb{R}^n$ defined by the inequalities $g_i \geq 0$, $i = 1, \dots, s$ is compact, $f \geq 0$ on K, and f has just finitely many zeros in K. Then the following are equivalent:*

1. *f lies in the preordering of $\mathbb{R}[\mathbf{x}]$ generated by $g_1, \dots, g_s$.*
2. *For each zero p of f in K, f lies in the preordering of the completion of $\mathbb{R}[\mathbf{x}]$ at p generated by $g_1, \dots, g_s$.*

In the two–dimensional case this allows one to show that certain finitely generated preorderings are saturated; see [12]. For example, Theorem 3 can be obtained by combining Theorem 4 with the following result for power series rings, using the Transfer Principle:

Theorem 5. *Suppose $f \in \mathbb{R}[[x, y]]$.*

1. *If $f \geq 0$ at each ordering of $\mathbb{R}((x, y))$ then f is a sum of squares in $\mathbb{R}[[x, y]]$.*
2. *If $f \geq 0$ at each ordering of $\mathbb{R}((x, y))$ satisfying $x > 0$ then f lies in the preordering of $\mathbb{R}[[x, y]]$ generated by x.*
3. *If $f \geq 0$ at each ordering of $\mathbb{R}((x, y))$ satisfying $x > 0$ and $y > 0$ then f lies in the preordering of $\mathbb{R}[[x, y]]$ generated by x and y.*

Proof. (1) is well-known. It can be proved using a modification of the analytic argument given in [2, Lem. 7a]. The proof shows, in fact, that f is a sum of two squares. See [6, Th. 1.6.3] for more details. (2) (resp., (3)) follows immediately from (1) by going to the extension ring $\mathbb{R}[[\sqrt{x}, y]]$ (resp., to the extension ring $\mathbb{R}[[\sqrt{x}, \sqrt{y}]]$). E.g., to prove (2), apply (1) to $\mathbb{R}[[\sqrt{x}, y]]$ to deduce $f = \sum f_i^2$, $f_i \in \mathbb{R}[[\sqrt{x}, y]]$. Decomposing $f_i = f_{i1} + f_{i2}\sqrt{x}$, $f_{ij} \in \mathbb{R}[x, y]]$, and expanding, yields $f = \sum f_{i1}^2 + \sum f_{i2}^2 x$. □

The main result of this paper is the following extension of Theorem 5.

Theorem 6. *Suppose $f \in \mathbb{R}[[x, y]]$ and n is a positive integer.*

1. *If $f \geq 0$ at each ordering of $\mathbb{R}((x, y))$ satisfying $y > 0$ and $x^{2n} - y > 0$ then f lies in the preordering of $\mathbb{R}[[x, y]]$ generated by y and $x^{2n} - y$.*
2. *If $f \geq 0$ at each ordering of $\mathbb{R}((x, y))$ satisfying $x > 0$, $y > 0$ and $x^n - y > 0$ then f lies in the preordering of $\mathbb{R}[[x, y]]$ generated by x, y and $x^n - y$.*

Remark 7. Suppose n is odd, $n \geq 3$. Then:

(i) For every ordering of $\mathbb{R}[[x, y]]$, $y \geq 0$ and $x^n - y \geq 0 \Rightarrow x \geq 0$, but x is not in the preordering of $\mathbb{R}[[x, y]]$ generated by y and $x^n - y$. This shows that an obvious attempt to strengthen Theorem 6 fails.

(ii) Similarly, for every ordering of $\mathbb{R}[[x, y]]$, $x^n - y^2 \geq 0 \Rightarrow x \geq 0$, but x is not in the preordering of $\mathbb{R}[[x, y]]$ generated by $x^n - y^2$.

Note: Going to $\mathbb{R}[[x, \sqrt{y}]]$, we see that assertions (i) and (ii) are essentially equivalent.

At the same time, there is no claim that Theorem 6 is the end of the story. It is conceivable that other similar results might exist. See the Concluding Remarks (Remark 10) for a systematic discussion.

We postpone the proof of Theorem 6 to Section 4. We now explain how Theorems 4, 5 and 6 can be combined to prove the following extension of Theorem 3.

Corollary 8. *Let $S = \{g_1, ..., g_s\}$ be irreducible polynomials in $\mathbb{R}[x, y]$. Suppose that $K = K_S \subseteq \mathbb{R}^2$ is compact, and for each boundary point p of K, either*

1. *there exists i such that p is a non-singular zero of g_i, and K is defined locally at p by the single inequality $g_i \geq 0$; or*
2. *there exists i, j such that p is a non-singular zero of g_i and g_j, g_i and g_j meet transversally at p, and K is defined locally at p by $g_i \geq 0$, $g_j \geq 0$; or*

(3) *there exists* i, j *such that* p *is a non-singular zero of* g_i *and* g_j, g_i *and* g_j *share a common tangent at* p *but do not cross each other at* p, *and* K *is described locally at* p *as the region between* $g_i = 0$ *and* $g_j = 0$; *or*

(4) *there exists* i, j, k *such that* p *is a non-singular zero of* g_i, g_j *and* g_k, g_i *and* g_j *share a common tangent at* p, g_i *and* g_k *meet transversally at* p, *and* K *is described locally at* p *as the part of the region between* $g_i = 0$ *and* $g_j = 0$ *defined by* $g_k \geq 0$.

Then the preordering of $\mathbb{R}[x, y]$ *generated by* $g_1, \ldots, g_s$ *is saturated.*

Proof. Let T denote the preordering of $\mathbb{R}[x, y]$ generated by $g_1, \ldots, g_s$. We wish to show that $f \in \mathbb{R}[x, y]$, $f \geq 0$ on $K \Rightarrow f \in T$. We may assume $K \neq \emptyset$, $f \neq 0$. The hypothesis implies, in particular, that K is the closure of its interior. This allows us to reduce further to the case where f is square-free and $g_i \nmid f$ for each i. In this situation, f has only finitely many zeros in K, so Theorem 4 applies, i.e., to show $f \in T$, it suffices to show that, for each zero p of f in K, f lies in the preordering of the completion of $\mathbb{R}[x, y]$ at p generated by $g_1, \ldots, g_s$. If p is an interior point of K this follows from Theorem 5(1). If p is a boundary point of K satisfying (1) (resp., (2), resp., (3), resp., (4)) then it follows from Theorem 5(2) (resp., Theorem 5(3), resp., Theorem 6(1), resp., Theorem 6(2)). We use the Transfer Principle and apply Theorems 5 and 6 with $x = \overline{x}$, $y = \overline{y}$, where $\overline{x}, \overline{y}$ are suitably chosen local parameters at p. If p is an interior point of K we choose $\overline{x} = x - a$, $\overline{y} = y - b$ where $p = (a, b)$. In case (1), we choose local parameters $\overline{x}$, $\overline{y}$ with $\overline{x} = g_i$. In case (2), we choose local parameters $\overline{x}$, $\overline{y}$ with $\overline{x} = g_i$, $\overline{y} = g_j$. In case (3), choose local parameters $\overline{x}$, g_i. By the Preparation Theorem [14, Cor. 1, p. 145], $hg_j = g_i + \overline{x}^n k$ for some unit h, some $n \geq 1$ and some unit $k \in \mathbb{R}[[\overline{x}]]$. Then $sg_i + tg_j = \overline{x}^n$ where $s = -\frac{1}{k}$ and $t = \frac{h}{k}$. By the geometry of the situation, the units s, t are positive units and n is even. Take $\overline{y} = sg_i$, so $\overline{x}^n - \overline{y} = tg_j$, and apply Theorem 6(1). In case (4) choose local parameters $\overline{x}$, g_i with $\overline{x} = g_k$. As before, this yields $sg_i + tg_j = \overline{x}^n$ for some units s, t and some $n \geq 1$. By the geometry of the situation, s, t are positive units. Take $\overline{y} = sg_i$, so $\overline{x}^n - \overline{y} = tg_j$, and apply Theorem 6(2). ☐

3. APPLICATIONS TO EQUIVARIANT SATURATED PREORDERINGS

In the following, we fix a group G together with $\phi : G \to \mathrm{GL}_n(\mathbb{R})$ a linear representation. We say that a subset $K \subseteq \mathbb{R}^n$ is G–**invariant** if $\phi(g)(K) \subseteq K$ for every $g \in G$. We can use ϕ to define an action of G on the polynomial ring $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$: given $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}], g \in G$ acts on $p(\mathbf{x})$ by $p^g(\mathbf{x}) = p(\phi(g)^{-1}\mathbf{x})$. The polynomial $p(\mathbf{x})$ is said to be G-invariant if $p^g(\mathbf{x}) = p(\mathbf{x})$ for all $g \in G$. The set of all G-invariant polynomials will be denoted by $\mathbb{R}[\mathbf{x}]^G$.

We now introduce the notion of G-saturation: We say that a preordering T_S is G-**saturated** if every G-invariant polynomial which is non-negative on K_S belongs to T_S. If T_S is saturated, then T_S is also G-saturated but the converse is false (see Example 9).

We now look at a two–dimensional example. Let $G = \{a, b | a^n = b^2 = (ab)^2 = 1\} = D_n$ be the n-th dihedral group and suppose that it acts on $\mathbb{R}^2$ and $\mathbb{R}[x, y]$ in a "standard way". Clearly, $\mathbb{R}[x, y]^G$ is an $\mathbb{R}$-algebra containing

$$u(x, y) = x^2 + y^2 \text{ and } v(x, y) = \mathrm{re}(x + iy)^n = x^n - \binom{n}{2} x^{n-2} y^2 + \ldots$$

It can be shown that u, v are algebraically independent and that they generate $\mathbb{R}[x, y]^G$. Hence, the mapping $\tilde{\pi}\colon f(u, v) \mapsto f(u(x, y), v(x, y))$ from $\mathbb{R}[u, v]$ to $\mathbb{R}[x, y]^G$ is an isomorphism. On the other hand, the mapping $\pi\colon (x, y) \mapsto (u(x, y), v(x, y))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is not onto. It is easy to see that

$$\pi(\mathbb{R}^2) = \{(u, v) | u \geq 0, -u^{n/2} \leq v \leq u^{n/2}\}.$$

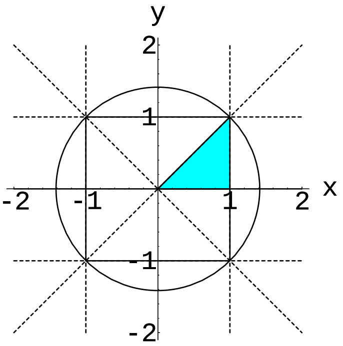
The mapping π is not one-to-one either. It can be shown that two points have the same image if and only if they lie in the same G-orbit. We call π the **orbit map** and $\pi(\mathbb{R}^2)$ the **orbit space**.

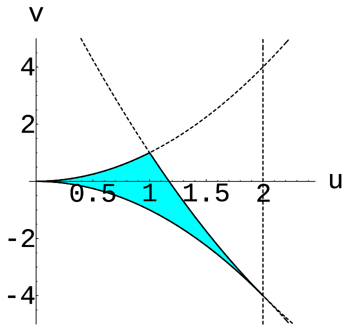
Let S be a finite subset of $\mathbb{R}[x, y]$ such that the set K_S is G-invariant. It is well-known (see e.g. [3, Proposition 3.15]) that there exists a finite subset S' of $\mathbb{R}[x, y]^G$ such that $K_S = K_{S'}$. It can be obtained from S by replacing each element $f \in S$ by all elementary symmetric polynomials in all *distinct* G-conjugates of f. Note that $T_{S'} \subseteq T_S$, but this inclusion can be strict (see Example 9). The new representation is instrumental for the computation of the image of $K_S = K_{S'}$ in the orbit space. Clearly, $\pi(K_{S'})$ is a basic closed semialgebraic set generated by $\tilde{\pi}^{-1}(S')$ and the generators of the orbit space (u and $u^{n/2} \pm v$ for n even, $u^n - v^2$ for n odd), i.e.

$$\pi(K_{S'}) = K_{\tilde{S}'} \text{ where } \tilde{S}' = \tilde{\pi}^{-1}(S') \cup \{u^{n/2} \pm v\} \text{ or } \tilde{S}' = \tilde{\pi}^{-1}(S') \cup \{u^n - v^2\}.$$

A natural question arises: when is $T_{S'} \subseteq \mathbb{R}[x, y]$ G-saturated? It is straightforward to verify that this is the case if and only if $T_{\tilde{S}'} \subseteq \mathbb{R}[u, v]$ is saturated. We shall exploit this observation in the following example, where we provide an affirmative answer in the case of the unit square. We cannot however answer the question in all generality (see the discussion in Remarks 11 and 12 of the Concluding Remarks).

Example 9. The unit square is generated by $S = \{1 - x, 1 + x, 1 - y, 1 + y\}$. It is a D_4-invariant set. By the discussion above, $K_S = K_{S'}$ where $S' = \{2 - x^2 - y^2, (1 - x^2)(1 - y^2)\} \subset \mathbb{R}[x, y]^{D_4}$ (picture on the left contains one point from each orbit). Note that T_S is saturated by [12]. On the other hand, it can be easily verified that $1 - x \notin T_{S'}$, hence $T_{S'}$ is not saturated. We want to determine whether, nevertheless, $T_{S'}$ is D_4-saturated. Consider $\pi(K_{S'}) = K_{\tilde{S}'}$ where $\tilde{S}' = \{2 - u, 1 - u + \frac{1}{8}(u^2 - v), u^2 - v, u^2 + v\}$ (picture on the right.)





Clearly, $\tilde{S}'$ satisfies the conditions of Corollary 8, hence $T_{\tilde{S}'}$ is D_4-saturated. It follows that $T_{S'}$ is D_4-saturated as required.

4. Proof of Theorem 6

Assertion (2) follows from assertion (1), by going to the extension ring $\mathbb{R}[[\sqrt{x}, y]]$, so it suffices to prove (1). We can assume $f \neq 0$. We know $\mathbb{R}[[x, y]]$ is a UFD [14, Th. 6, p. 148]. Factor f into irreducibles in $\mathbb{R}[[x, y]]$. Using the Preparation Theorem, we can assume the factorization has the form

$$f = ux^m g = ux^m \prod_{i=1}^{\ell} p_i^{m_i}$$

where u is a unit and each p_i is a monic polynomial in y with coefficients in $\mathbb{R}[[x]]$, with all coefficients except the leading coefficient in the maximal ideal of $\mathbb{R}[[x]]$. We can reduce to the case where $m = 0$ or 1 and g has no repeated irreducible factors. Since $\pm u$ is a square in $\mathbb{R}[[x, y]]$, we can assume further that $u = \pm 1$.

Since y and $x^{2n} - y$ are obviously in the preordering generated by y and $x^{2n} - y$, we can assume $y \nmid g$ and $y - x^{2n} \nmid g$. More generally, if g has an irreducible factor p which has constant sign on the set $y > 0$ in the real spectrum (see [1]) of $\mathbb{R}((x, y))$ then, by part (2) of Theorem 5, $\pm p$ is in the preordering generated by y. Similarly, if p has constant sign on the set $x^{2n} > y$ in the real spectrum of $\mathbb{R}((x,y))$ then, by part (2) of Theorem 5 (using the fact that $\mathbb{R}[[x,y]] = \mathbb{R}[[x, x^{2n} - y]]$), $\pm p$ is in the preordering generated by $x^{2n} - y$. Consequently, we can assume that g has no such irreducible factors.

Fix an irreducible factor p of g and consider the discrete valuation on $\mathbb{R}((x,y))$ with associated valuation ring $\mathbb{R}[[x,y]]_{(p)}$. The residue field is $L = \text{qf}\frac{\mathbb{R}[[x,y]]}{(p)} = \frac{\mathbb{R}((x))[y]}{(p)}$ [14, Th. 6, p. 148]. Set $\overline{y} = y + (p)$. Since $p \neq y$, $p \neq y - x^{2n}$, we know that $\overline{y} \neq 0$, $\overline{y} \neq x^{2n}$. L is a finite extension of the complete discrete valued field $\mathbb{R}((x))$ so it either has no orderings (if the residue field is $\mathbb{C}$) or two orderings (if the residue field is $\mathbb{R}$).

Claim 1: L has no ordering satisfying $0 < \overline{y} < x^{2n}$. Otherwise, pulling this ordering back to $\mathbb{R}((x,y))$ using Baer-Krull, yields two orderings on $\mathbb{R}((x,y))$ satisfying $0 < y < x^{2n}$, one with $p > 0$ and one with $p < 0$. Since an irreducible factor q of f different from p has the same sign at each of these two orderings, and since p has multiplicity 1 in f, one of these two orderings must make $f < 0$. This contradicts our assumption and proves the claim.

Claim 2: L has an ordering satisfying $\overline{y} > x^{2n}$ and also an ordering satisfying $\overline{y} < 0$. By assumption p is not always positive on the set $y > 0$ in the real spectrum of $\mathbb{R}((x,y))$, so there exists an ordering of $\mathbb{R}((x,y))$, with real closure R say, with $y > 0$ and $p(y) < 0$, so the polynomial p has a root $a > y$ in R. Then $\overline{y} \mapsto a$ defines an $\mathbb{R}((x))$-embedding of L into R, so L has an ordering satisfying $\overline{y} > 0$, i.e., $\overline{y} > x^{2n}$. We prove the second assertion when $\deg(p)$ is odd. The proof when $\deg(p)$ is even is similar. By assumption p is not always negative on the set $x^{2n} > y$ in the real spectrum of $\mathbb{R}((x,y))$, so there exists an ordering of $\mathbb{R}((x,y))$ with real closure R say, with $y < x^{2n}$ and $p(y) > 0$, so the polynomial p has a root $a < y$ in R. Then $\overline{y} \mapsto a$ defines an $\mathbb{R}((x))$-embedding of L into R, so L has an ordering satisfying $\overline{y} < x^{2n}$, i.e., $\overline{y} < 0$.

Denote the valuation on L by v. Since $p(\overline{y}) = 0$ we see that $v(\overline{y}) > 0$. Since L has an ordering satisfying $\overline{y} > x^{2n}$, it follows that $v(\overline{y}) \leq v(x^{2n})$. At the same time, $v(\overline{y}) = v(x^{2n})$ is not possible. (If $v(\overline{y}) = v(x^{2n})$ then $\overline{y} = ux^{2n}$, u a unit. Since $\overline{y}$ is positive at one ordering and negative at the other, the same would be true for u, which is not possible.) Thus $0 < v(\overline{y}) < v(x^{2n})$.

Of course, since the various roots a of p in the algebraic closure of $\mathbb{R}((x))$ are conjugate to $\overline{y}$ over $\mathbb{R}((x))$, they all have the same value $v(a) = v(\overline{y})$.

Write $f = \pm x^m p_1 \dots p_\ell$ where the p_i are irreducible, $p_i = \sum_{j=0}^{k_i} b_{ij} y^j$, $b_{ik_i} = 1$, $v(b_{i0}) = k_i v(a_i)$, $v(b_{ij}) \geq (k_i - j)v(a_i)$, where a_i is a fixed root of p_i. We know $0 < v(a_i) < v(x^{2n})$. Decompose f as

$$f = f(0) + \sum_{\underline{j} \neq (0,\dots,0)} \pm x^m b_{\underline{j}} y^{j_1 + \dots + j_\ell}$$

where $\underline{j} := (j_1, \dots, j_\ell)$, $b_{\underline{j}} := b_{1j_1} \dots b_{\ell j_\ell}$ and $f(0) := \pm x^m b_{10} \dots b_{\ell 0}$.

Claim 3: $f(0)$ is positive at both orderings of $\mathbb{R}((x))$, i.e., $f(0)$ is a square in $\mathbb{R}[[x]]$. Suppose to the contrary that $f(0)$ is negative at one of the orderings of $\mathbb{R}((x))$. Consider the discrete valuation on $\mathbb{R}((x, y))$ with valuation ring $\mathbb{R}[[x, y]]_{(y)}$ and residue field $\mathbb{R}((x))$. Pulling the culprit ordering of $\mathbb{R}((x))$ back to $\mathbb{R}((x, y))$, using Baer-Krull, yields two orderings of $\mathbb{R}((x, y))$, one of which satisfies $x^{2n} > y > 0$ and $f < 0$. This is a contradiction.

We write each term $\pm x^m b_{\underline{j}} y^{j_1 + \dots + j_\ell}$, $\underline{j} \neq (0, \dots, 0)$ in (1) as

$$(c_{\underline{j}} \pm x^m b_{\underline{j}}) y^{j_1 + \dots + j_\ell} + c_{\underline{j}} (x^{2n(j_1 + \dots + j_\ell)} - y^{j_1 + \dots + j_\ell}) - c_{\underline{j}} x^{2n(j_1 + \dots + j_\ell)}.$$

Factoring in the obvious way, we see that $x^{2n(j_1 + \dots + j_\ell)} - y^{j_1 + \dots + j_\ell}$ lies in the preordering generated by $x^{2n} - y$ and y. To complete the proof, it suffices to show we can choose the elements $c_{\underline{j}} \in \mathbb{R}[[x]]$, $\underline{j} \neq (0, \dots, 0)$, so that

$$c_{\underline{j}} \pm x^m b_{\underline{j}}, \ c_{\underline{j}} \ \text{ and } \ f(0) - \sum_{\underline{j} \neq (0,\dots,0)} c_{\underline{j}} x^{2n(j_1 + \dots + j_\ell)}$$

are squares in $\mathbb{R}[[x]]$. Since $\underline{j} \neq (0, \dots, 0)$,

$$\begin{aligned} v(x^m b_{\underline{j}}) &= v(x^m) + \sum_i v(b_{ij_i}) \\ &\geq v(x^m) + \sum_i (k_i - j_i) v(a_i) \\ &= v(x^m) + \sum_i k_i v(a_i) - \sum_i j_i v(a_i) \\ &> v(x^m) + \sum_i k_i v(a_i) - \sum_i j_i v(x^{2n}) \\ &= v(x^m) + \sum_i v(b_{i0}) - \sum_i j_i v(x^{2n}) \\ &= v\Big(\frac{f(0)}{x^{2n(j_1 + \dots + j_\ell)}}\Big). \end{aligned}$$

We choose the $c_{\underline{j}}$ as follows: If $\underline{j} \neq (k_1, \dots, k_\ell)$ or $\underline{j} = (k_1, \dots, k_\ell)$ and $m = 1$, then $x^m b_{\underline{j}}$ has positive value. In this case, we choose $c_{\underline{j}}$ with small positive lowest coefficient and with

$$v(c_{\underline{j}}) = \max\{v(\frac{f(0)}{x^{2n(j_1+\cdots+j_\ell)}}), 0\}.$$

In the remaining case, where $m = 0$ and $\underline{j} = (k_1, \ldots, k_\ell)$, $b_{ij_i} = 1$, $i = 1, \ldots, \ell$, and we choose $c_{\underline{j}} = 1$. The point is, with this choice of $c_{\underline{j}}$, for each $\underline{j} \neq (0, \ldots, 0)$, either $c_{\underline{j}} x^{2n(j_1+\cdots+j_\ell)}$ has larger value than $f(0)$ or, it has the same value as $f(0)$, but its lowest coefficient is small.

5. Concluding Remarks

We start this Section with a few remarks which place what we have been doing (in Theorem 6) in a more general context, and which, at the same time, indicate that there are two additional questions one should be considering (see (i) and (j) below.)

Remark 10. Let $\mathfrak{m}$ denote the unique maximal ideal of $\mathbb{R}[[x,y]]$, i.e., the ideal generated by x and y. For $f \in \mathbb{R}[[x,y]]$, let $\operatorname{ord}(f) :=$ the least i such that $f \notin \mathfrak{m}^{i+1}$. Equivalently, if $f = \sum_{i,j\geq 0} a_{ij}x^i y^j$, then $\operatorname{ord}(f) = \min\{i+j \mid a_{ij} \neq 0\}$. $\operatorname{ord}(0) := \infty$. ord extends to a discrete valuation on $\mathbb{R}((x,y))$.

We often make a change in variables. This means we replace x, y by u, v where u, v are new generators for the ideal $\mathfrak{m}$, i.e., u, v are elements of $\mathfrak{m}$ which meet transversally at $(0,0)$, i.e., $u \equiv ax + by \mod \mathfrak{m}^2$, $v \equiv cx + dy \mod \mathfrak{m}^2$, $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$. In this situation, $\mathbb{R}[[x,y]] = \mathbb{R}[[u,v]]$.

Fix a finite subset S of $\mathbb{R}[[x,y]]$ and let $T :=$ the preordering of $\mathbb{R}[[x,y]]$ generated by S. Below, we denote by $\operatorname{Sper} A$ the real spectrum of a ring A (see [1]). Let $K := \{P \in \operatorname{Sper}\mathbb{R}[[x,y]] \mid S \subseteq P\}$. The **saturation** of T is $\tilde{T} := \cap_{P\in K} P$. T is **saturated** iff $T = \tilde{T}$.

In view Tarski's Transfer Principle and Theorem 4, one would like to know when T is saturated. This seems to be a difficult question. Making a change in variables, replacing x, y by $x + cy, y$ for a suitable $c \in \mathbb{R}$, we can assume, for each non-zero $g \in S$, if $\ell = \operatorname{ord}(g)$, then y^ℓ actually appears in g. Each unit in $\mathbb{R}[[x,y]]$ is either the square of a unit or minus the square of a unit. Using this and the Preparation Theorem, multiplying each non-zero $g \in S$ by the square of a suitable unit, we may assume each $g \in S$ has the form

$$g = 0 \text{ (if } \operatorname{ord}(g) = \infty)$$

$$g = \pm 1 \text{ (if } \operatorname{ord}(g) = 0)$$

$$g = \pm(y^\ell - \textstyle\sum_{i=1}^{\ell} a_i(x) y^{\ell-i})$$

where $a_i(x) \in \mathbb{R}[[x]]$, $\operatorname{ord}(a_i(x)) \geq i$, $i = 1, \dots, \ell$ (if $\operatorname{ord}(g) = \ell \geq 1$). 0 and 1 do not contribute to T, so there is no harm in assuming $0, 1 \notin S$. If $-1 \in S$ then $T = \mathbb{R}[[x, y]]$, which is trivially saturated, so we may assume also that $-1 \notin S$. Let us restrict now to the case where each non-zero $g \in S$ has order ≤ 1 (for an example when this assumption fails, see Remark 11). In this case the above analysis reduces us to the case where each $g \in S$ has order 1, i.e., has the form

$$g = \pm(y - a(x)),\ a(x) \in \mathbb{R}[[x]],\ a(0) = 0.$$

Not all the inequalities $g \geq 0$, $g \in S$ may be needed to describe K. In fact, an easy argument shows that there is always a subset S' of S with $|S'| \leq 4$ describing K.

Decompose S as $S^+ \cup S^-$ by putting elements of S of the form $y - a(x)$ in S^+ and elements of S of the form $-(y - a(x))$ in S^-. Of course, one or both of S^+, S^- may be empty. Consider the subring $\mathbb{R}[[x]]$ of $\mathbb{R}[[x, y]]$. This subring has exactly 3 orderings (one with $x > 0$, one with $x = 0$ and one with $x < 0$). Let $\operatorname{Sper} \mathbb{R}[[x, y]] = \operatorname{Sper}_+ \cup \operatorname{Sper}_0 \cup \operatorname{Sper}_-$ be the corresponding decomposition of $\operatorname{Sper} \mathbb{R}[[x, y]]$, i.e., Sper_+ is the set of orderings of $\mathbb{R}[[x, y]]$ making $x > 0$, etc.. Since $\mathbb{R}[[x, y]]/(x) \cong \mathbb{R}[[y]]$, Sper_0 has just 3 elements. On Sper_+, the elements of S^+ are totally ordered. Let $y - a_1(x)$ be the element of S^+ which is smallest on Sper_+. Similarly, let $y - a_2(x)$ be the element of S^+ which is smallest on Sper_-. $-(y - b_1(x))$ and $-(y - b_2(x))$ are defined similarly, but using S^- now, instead of S^+. Take $S' = \{y - a_1(x), y - a_2(x), -(y - b_1(x)), -(y - b_2(x)\}$. We are assuming here that S^+ and S^- are both non-empty. If $S^- = \emptyset$, we get $S' = \{y - a_1(x), y - a_2(x)\}$. If $S^+ = \emptyset$, we get $S' = \{-(y - b_1(x)), -(y - b_2(x))\}$. If S^+ and S^- are both empty (i.e., $S = \emptyset$), we get $S' = \emptyset$.

Using this, and considering the various possibilities, we see that K is either the "whole plane", a "half-plane", a "wedge", "two wedges" or a "point"(the unique ordering with support $\mathfrak{m}$).

In the "whole plane" case, $S = \emptyset$ and K is all of $\operatorname{Sper} \mathbb{R}[[x, y]]$. In the "half-plane" case, either $S^- = \emptyset$ and $a_1(x) = a_2(x)$ or $S^+ = \emptyset$ and $b_1(x) = b_2(x)$. Let $K_+ := K \cap \operatorname{Sper}_+$, $K_- := K \cap \operatorname{Sper}_-$. In the "wedge" case, either S^+ and S^- are non-empty and one of K_+, K_- is empty and the other is not, or $S^- = \emptyset$ and $a_1(x) \neq a_2(x)$, or $S^+ = \emptyset$ and $b_1(x) \neq b_2(x)$. In the "two wedge" case, S^+ and S^- are non-empty and K_+ and K_- are non-empty. In the "point" case, S^+ and S^- are non-empty, and K_+ and K_- are empty.

(a) In the "whole plane" case, $T = \sum \mathbb{R}[[x, y]]^2$, which is saturated, by Theorem 5(1).

(b) In the "half-plane" case, making an appropriate change in variables, K is defined by $y \geq 0$, $y \in S$. In this case we know T is saturated, by Theorem 5(2).

In the "wedge" case, making an appropriate change in variables, we have either

(c) K is defined by $x \geq 0$, $y \geq 0$, and $x, y \in S$; or
(d) K is defined by $x \geq 0$ and $x^k \geq y \geq 0$, where $k \geq 2$ and $y, x^k - y \in S$ (note: we are not assuming $x \in S$); or
(e) K is defined by $x \geq 0$ and $y = 0$, and $y, -y \in S$ (again, we are not assuming $x \in S$); or
(f) K is defined by $y \geq 0$ and $y \geq x^k$, where k is odd, $k \geq 3$ and $y, y - x^k \in S$.

In case (c) we know T is saturated, by Theorem 5(3).

In case (d) we show that T is saturated iff there is some $g \in S$ meeting y transversally at $(0,0)$ (using Theorem 6(2) and the following additional argument): 1. Assume T is saturated, $g_1, \dots, g_s$ are generators for T. We *claim* one of the g_i is transversal to y. Assume not. Let $\mathfrak{a}$ denote the ideal of $\mathbb{R}[[x,y]]$ generated by y and x^2, i.e., $\mathfrak{a} = (y) + \mathfrak{m}$. By assumption each g_i belongs to $\mathfrak{a}$. Since $x \geq 0$ on K, $x \in T$, so x has a presentation as a sum of elements $\sigma g_1^{e_1} \dots g_s^{e_s}$, $\sigma \in \sum \mathbb{R}[[x,y]]^2$, $e_i \in \{0,1\}$. All terms with $(e_1, \dots, e_s) \neq (0, \dots, 0)$ belong to $\mathfrak{a}$. Thus x has a presentation $x \equiv \sigma \mod \mathfrak{a}$, σ a sum of squares, say $\sigma = \sum f_i^2$. If f_i has order 0 for some i, then σ has order 0, so $x - \sigma$ has order 0, contradicting $x - \sigma \in \mathfrak{a}$. If all f_i have order ≥ 1, then σ has order ≥ 2, so $x \equiv x - \sigma \equiv 0 \mod \mathfrak{a}$, again a contradiction.

2. Suppose there exists $g \in T$ transversal to y, i.e., $g = rx + sy +$ terms of degree ≥ 2, $r, s \in \mathbb{R}$, $r \neq 0$. Because $g \geq 0$ on K and from the way K is described we see that $r > 0$. Scaling, we may assume $r = 1$. By the Preparation Theorem, $ug = x - a(y)$ for some unit u and some $a(y) \in \mathbb{R}[[y]]$ of order ≥ 1. Comparing coefficients of x, we see that u is a positive unit. Scaling, we may assume $g = x - a(y)$. Let $h = yu_1$ where $u_1 := 1 - \sum_{k=1}^n \begin{pmatrix} n \\ k \end{pmatrix} g^{n-k} \frac{a(y)^k}{y}$. Then $x^n - y = (g + a(y))^n - y = g^n + \sum_{k=1}^n \begin{pmatrix} n \\ k \end{pmatrix} g^{n-k} a(y)^k - y = g^n - h$. Also, u_1 is a positive unit, so $h \in T$. Thus $\mathbb{R}[[x,y]] = \mathbb{R}[[g,h]]$ and $g, h, g^n - h \in T$. By Th. 6(2) the preordering generated by $g, h, g^n - h$ is saturated. Since this preordering is contained in T and defines the same set K that T does, we see that T is saturated.

Case (e) is the same as case (d), only simpler.

In case (f) the question is open. If the preordering generated by y and $y - x^k$ is saturated then T is also saturated in this case.

In the "double wedge" case, making an appropriate change in variables, we have either:

(g) K is defined by $x^k \geq y \geq 0$, where k is even, $k \geq 2$ and $y, x^k - y \in S$; or

(h) K is defined by $y = 0$, where $y, -y \in S$; or

(i) K is defined by $y \geq 0$, $y \geq x^k$ and $x^\ell \geq y$, where k is odd, ℓ is even, $k > \ell \geq 2$ and $y, y - x^k, x^\ell - y \in S$; or

(j) K is defined by $y \geq 0$, $y \geq x^k$, $x^\ell \geq y$ and $x^\ell(1 + a(x)) \geq y$, where k is odd, ℓ is even, $k > \ell \geq 2$, $a(x) \in \mathbb{R}[[x]]$, $a(0) = 0$ and $y, y - x^k, x^\ell - y, x^\ell(1 + a(x)) - y \in S$.

In case (g), T is saturated, by Theorem 6(1). In case (h), T is obviously saturated. In case (i) (resp., (j)), the question is open. If the preordering generated by $y, y - x^k, x^\ell - y$ (resp., by $y, y - x^k, x^\ell - y, x^\ell(1 + a(x)) - y$) is saturated then T is also saturated in this case.

(k) in the "point" case one checks easily that T is saturated iff the cone generated by the image of S in the vector space $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{R}x \oplus \mathbb{R}y$ is all of $\mathfrak{m}/\mathfrak{m}^2$.

Recall that the above discussion was under the assumption that each nonzero $g \in S$ is irreducible and has order ≤ 1. The case when singular points appear seems to be difficult. For example, the following problem is open: Is it true that if $f \geq 0$ holds for every ordering of $\mathbb{R}[[x, y]]$ satisfying $y^2 - x^n \geq 0$ then f is in the preordering generated by $y^2 - x^n$?

The case when some g_i is reducible is difficult as well. Note that these cases are encountered in the context of D_n-saturation discussed at the end of Section 3 (below, we continue using the notation and terminology of Section 3):

Remark 11. For odd n, the orbit space $\pi(\mathbb{R}^2)$ has a cusp at the origin. Thus for every D_n-invariant basic closed semialgebraic set containing zero, we cannot, using our main corollary, decide D_n-saturation of the corresponding preordering.

Remark 12. Suppose that K_S is D_n-invariant and $p \in \partial K_S$ (=the boundary of K_S). If S satisfies at p one of the situations from the Corollary, then we would like to know whether $\tilde{S}'$ also satisfies at $\pi(p)$ one of the situations from the Corollary. We must discuss the following issues:

(i) What does π do to the angle of K_S at p?

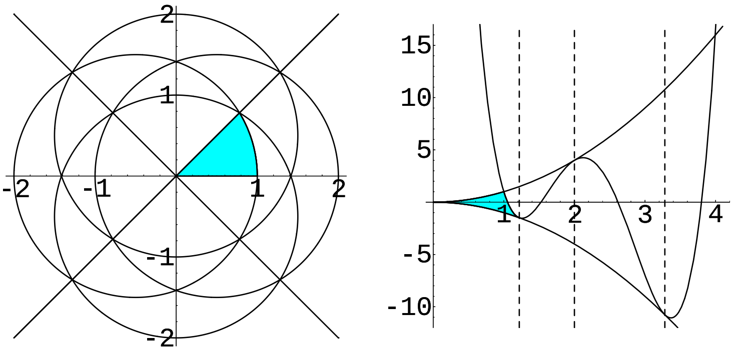
(ii) Is regularity at p and irreducibility preserved when we pass from S to $\tilde{S}'$?

Write $\Delta = \{(r,\phi) | 0 \leq \phi \leq \pi/n\}$ and note that the orbit map π is nonsingular on Δ^0 (=the interior of Δ) and singular on $\partial\Delta$. Write $K = K_S \cap \Delta$ and note that $\pi(K_S) = \pi(K)$.

Pick a point $p \in \partial K$ and write α for the angle of K at p and β for the angle of $\pi(K)$ at $\pi(p)$. The following observations can be made:

(1) if $p \in \Delta^0$ and $\alpha \neq 0$ then $\beta \neq 0$,
(2) if $p \in \Delta^0$ and $\alpha = 0$ then $\beta = 0$,
(3) if $p \in \partial\Delta$ and $0 \leq \alpha < \pi/2$ then $\beta = 0$,
(4) if $p \in \partial\Delta$ and $\pi/2 < \alpha \leq \pi$ then $\beta = \pi$,
(5) if $p \in \partial\Delta$ and $\alpha = \pi/2$ then $0 < \beta < \pi$.

In Cases (1) and (2), the nonsingularty of π ensures that it preserves the situations from the Corollary, including irreducibility and regularity. In Cases (3)–(5) we can't say much about the preservation of regularity and irreducibility (it must be verified on a case by case basis). In general, the irreducibility of f does not imply the irreducibility of $\tilde{\pi}^{-1}(f_j)$. E.g. for $f = 2 - x^2 - y^2 + x$, we get $\tilde{\pi}^{-1}(f_1) = 2 - u$, $\tilde{\pi}^{-1}(f_2) = 24 - 25u + 6u^2$, $\tilde{\pi}^{-1}(f_3) = 32 - 52u + 26u^2 - 4u^3$, $\tilde{\pi}^{-1}(f_4) = 16 - 36u + 28u^2 - 9u^3 + u^4 + (u^2 - v)/8$, see picture:

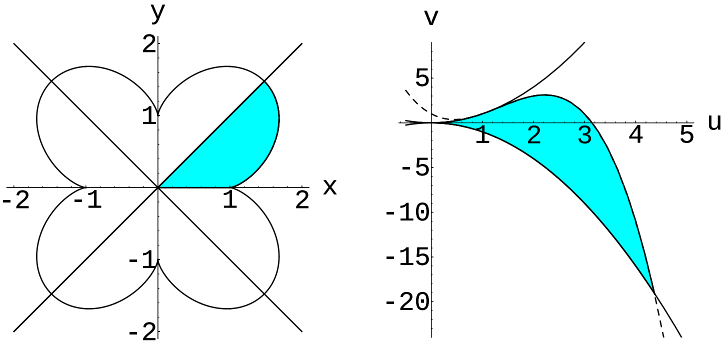


Note that $\tilde{\pi}^{-1}(f_4) = 0$ touches $v = -u^2$ at $u_{1,2} = (9 \pm \sqrt{17})/4$ and that $\tilde{\pi}^{-1}(f_3) = (2-u)(u_1 - u)(u_2 - u) = 0$ (dashed) is the only candidate for the transversal section at the right-most point.

In Case (3), we need a transversal section at $\pi(p)$. It suffices to have an $g \in S'$ that passes through p and is nonsingular at p. Because of the symmetry such a g meets $\partial\Delta$ at right angle, hence $\tilde{\pi}^{-1}(g)$ is the desired transversal section by Case (5). Unfortunately, we don't know in advance whether such g exists.

Case (4) is not accessible at the moment because our Corollary 8 does not cover "straight angles". An example of this case is given by

$p = (1,0)$ and $S = \{f\}$ where $f(x,y) = x^4 - 6x^2y^2 + y^4 - (x^2+y^2)^2 + (x^2+y^2-1)^3\}$.



However, $f(x,y)$ is singular at p. Note that Case (4) cannot appear if $S = \{g_1, \dots, g_m\}$ where $g_1, \dots g_m$ are regular at p. Case (5) is already covered by Scheiderer's result Theorem 3 (once irreducibility and regularity are dealt with.)

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