Mathematics 101.3 Final Examination Solutions

December 16, 1999

PART I: The answers to all questions are the digits in **ANSWER SET I:**

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4	(F) 5	(G) 6	(H) 7	(I) 8	(J) 9

If $\frac{8}{9} - \frac{6}{7}$ is written in its simplest form as $\frac{a}{10b+c}$, where a, b, and c are digits, then

(1) $a =$ **(2)** $b =$ **(3)** $c =$

Solution: $\frac{8}{9} - \frac{6}{7} = \frac{8 \times 7 - 6 \times 9}{9 \times 7} = \frac{56 - 54}{63} = \frac{2}{6 \times 10 + 3}$, so

(1) $a =$ **2** **(2)** $b =$ **6** **(3)** $c =$ **3**

$x^2 + 10x + 33$ is to be written in the form $(x + b)^2 + c$ by completing squares.

We must have: **(4)** $b =$ **(5)** $c =$

Solution: $x^2+10x+33 = x^2+2(5)x+33 = x^2+2(5)x+5^2-5^2+33 = (x+5)^2+8$, so

(4) $b =$ **5** **(5)** $c =$ **8**

$7x^2 + 112x + 490$ is to be written in the form $7[(x + b)^2 + c]$ by completing squares.

We must have: **(6)** $b =$ **(7)** $c =$

Solution: $7x^2 + 112x + 490 = 7[x^2 + 16x + 70] = 7[x^2 + 2(8)x + 70] =$

$7[x^2 + 2(8)x + 8^2 - 8^2 + 70] = 7[(x + 8)^2 + 6]$, so

(6) $b =$ **8** **(7)** $c =$ **6**

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(8) If m is the slope of the line through the points $(10, 15)$ and $(20, 75)$, then m is:

Solution: $m = \dfrac{75 - 15}{20 - 10} = \dfrac{60}{10} = 6$, so **6**

The roots of $4x^2 + 3x - 3 = 0$ in their simplest form are $\dfrac{-a \pm \sqrt{10b + c}}{d}$. We must have:

(9) $a =$ **(10)** $b =$ **(11)** $c =$ **(12)** $d =$

Solution: $\dfrac{-3 \pm \sqrt{3^2 - 4(4)(-3)}}{2(4)} = \dfrac{-3 \pm \sqrt{9 + 48}}{8} = \dfrac{-3 \pm \sqrt{57}}{8} = \dfrac{-3 \pm \sqrt{10(5) + 7}}{8}$, so

(9) $a =$ **3** **(10)** $b =$ **5** **(11)** $c =$ **7** **(12)** $d =$ **8**

$\dfrac{3\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}}$ can be simplified to the expression $a + \sqrt{10b + c}$, where a, b, and c are digits. Their values are: **(13)** $a =$ **(14)** $b =$ **(15)** $c =$

Solution:

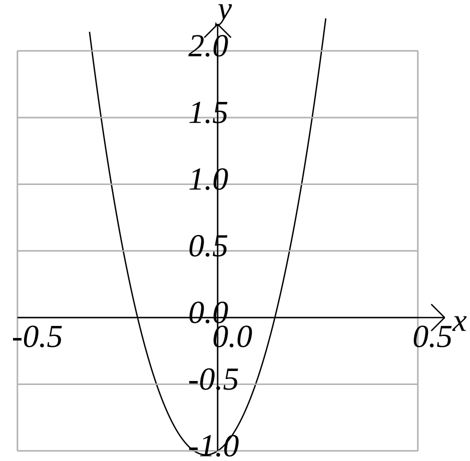
$$\frac{3\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} = \frac{3\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} + \sqrt{3}} = \frac{3\sqrt{7}\sqrt{7} + \sqrt{3}\sqrt{7} + 3\sqrt{7}\sqrt{3} + \sqrt{3}\sqrt{3}}{7 - 3} =$$

$$\frac{3(7) + \sqrt{21} + 3\sqrt{21} + 3}{4} = \frac{24 + 4\sqrt{21}}{4} = 6 + \sqrt{10(2) + 1}, \text{ so}$$

(13) $a =$ **6** **(14)** $b =$ **2** **(15)** $c =$ **1**

The polynomial $p(x) = 35x^4 + 2x^3 + 34x^2 + 2x - 1$ can be factored in the form $(x^2 + a^2)(bx + 1)(cx - 1)$, where a, b, and c are digits. Its graph is shown to the right.

We must have: **(16)** $a =$ **(17)** $b =$ **(18)** $c =$



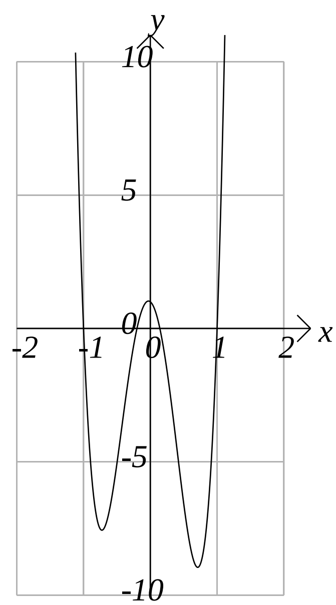
Solution:

Letting $x = 0$, we get $p(0) = -1$ and $(0^2 + a^2)(0x + 1)(c(0) - 1) = -a^2$, so we must have $a = 1$. The only possible rational roots are ± 1, $\pm\frac{1}{5}$, $\pm\frac{1}{7}$, and $\pm\frac{1}{35}$. The graph tells us that they are $-\frac{1}{5}$ and $\frac{1}{7}$, so we have $b = 5$ and $c = 7$:

(16) $a =$ **1** **(17)** $b =$ **5** **(18)** $c =$ **7**

The polynomial $p(x) = 35x^4 + 2x^3 - 36x^2 - 2x + 1$ can be factored in the form $(x^2 - a^2)(bx + 1)(cx - 1)$, where a, b, and c are digits. Its graph is shown to the right.

We must have: **(19)** $a =$ **(20)** $b =$ **(21)** $c =$



Solution:

Letting $x = 0$, we get $p(0) = 1$ and $(0^2 - a^2)(0x + 1)(c(0) - 1) = a^2$, so we must have $a = 1$. The only possible rational roots are ± 1, $\pm\frac{1}{5}$, $\pm\frac{1}{7}$, and $\pm\frac{1}{35}$. The graph tells us that they are -1, $-\frac{1}{5}$, $\frac{1}{7}$, and 1, so we again have $b = 5$ and $c = 7$:

(19) $a =$ **1** **(20)** $b =$ **5** **(21)** $c =$ **7**

If we solve the inequality $\left|\frac{1-x}{17}\right| < 1$, the solution is an interval of the form $(-(10+a), 10+b)$.

The values of a and b are: **(22)** $a =$ **(23)** $b =$

Solution:

The inequality is equivalent to

$|x-1| < 17$, the set of all numbers within 17 of 1, so the solution set is $(-16, 18)$. Thus we have:

(22) $a =$ **6** **(23)** $b =$ **8**

Consider the line perpendicular to the line $y = \frac{8x}{5} + 1$ which passes through the point $\left(\frac{40}{13}, \frac{40}{13}\right)$

(24) What is its x-intercept? **(25)** What is its y-intercept?

Solution:

The slope of the perpendicular line is $-\frac{5}{8}$, so its Point-Slope Form is

$y - \frac{40}{13} = -\frac{5}{8}\left(x - \frac{40}{13}\right)$. Multiplying by $8 \times 13 = 104$, we get

$104y - 320 = -5(15x - 40) = -65x + 200$ or

$104y + 65x = 520$ or

$\frac{104y}{520} + \frac{65x}{520} = 1$ or

$\frac{y}{5} + \frac{x}{8} = 1$, so we have:

(24) **8** **(25)** **5**

Evaluate the limit: **(26)** $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ if $f(x) = \dfrac{16}{(x+2)^2}$ and $x = -4$ **4**

Solution:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{16}{((x+h)+2)^2} - \frac{16}{(x+2)^2}}{h} = 16\lim_{h\to 0} \frac{(x+2)^2-((x+h)+2)^2}{((x+h)+2)^2(x+2)^2h} =$$

$$16\lim_{h\to 0} \frac{(x+2)^2-((x+2)+h)^2}{((x+h)+2)^2(x+2)^2h} = 16\lim_{h\to 0} \frac{(x+2)^2-((x+2)^2+2(x+2)h+h^2)}{((x+h)+2)^2(x+2)^2h} =$$

$$16\lim_{h\to 0} \frac{-(2(x+2)+h)}{((x+h)+2)^2(x+2)^2} = 16\frac{-(2(x+2)+0)}{((x+0)+2)^2(x+2)^2} =$$

$16\dfrac{-(2(-4+2))}{((-4+2)^4} = 16\dfrac{4}{((-2)^4} = 4$, so the answer is **4**

Evaluate the limit: **(27)** $\lim_{x\to 2} \dfrac{5x^2-5x-210}{5x-35}$

Solution:

$$\lim_{x\to 2} \frac{5x^2-5x-210}{5x-35} = \frac{5(2)^2-5(2)-210}{5(2)-35} = \frac{5(4)-10-210}{10-35} = \frac{-200}{-25} =$$ **8**

Evaluate the limit:

(28) $\lim_{x\to 6} \dfrac{x^2-8x+12}{x-6}$ **Solution:**

$$\lim_{x\to 6} \frac{x^2-8x+12}{x-6} = \lim_{x\to 6} \frac{(x-6)(x-2)}{x-6} = \lim_{x\to 6} x-2 = 6-2 =$$ **4**

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(29) The natural domain of the function $f(x) = \ln(1-7x^2)$ is of the form $\left(-\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{a}}\right)$. a =?

Solution: We must have $1-7x^2 > 0$ so $x^2 < \frac{1}{7}$. The answer is thus **7**

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(30) Let $f(x) = \dfrac{(x+1)^3}{(x+5)^3}$. Find $\dfrac{f'(-6)}{100}$

Solution: First rewrite the function $f(x) = \left(\dfrac{x+1}{x+5}\right)^3$, and then apply the Chain Rule followed by the Quotient Rule:

$$f'(x) = 3\left(\frac{x+1}{x+5}\right)^2\left(\frac{x+1}{x+5}\right)' = 3\left(\frac{x+1}{x+5}\right)^2\frac{(x+5)(x+1)' - (x+1)(x+5)'}{(x+5)^2} =$$

$$3\left(\frac{x+1}{x+5}\right)^2\frac{(x+5)-(x+1)}{(x+5)^2} = 3\left(\frac{x+1}{x+5}\right)^2\frac{4}{(x+5)^2}$$

Thus $f'(-6) = 3\left(\dfrac{-6+1}{-6+5}\right)^2\dfrac{4}{(-6+5)^2} = 3\left(\dfrac{-5}{-1}\right)^2\dfrac{4}{(-1)^2} = 300$

Thus $\dfrac{f'(-6)}{100} =$ **3**

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(31) The minimum value of $f(x) = 3x^2 - 18x + 30$ is:

Solution: $f'(x) = 6x - 18 = 0$ if $x = 3$. Since $f''(x) = 6 > 0$, this gives a minimum, $f(3) =$ **3**

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(32) Find the solution of the equation $2^{3x-1} = 256$:

Solution: Since $256 = 2^8$, we must have $3x - 1 = 8$, so $x =$ **3**

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(33) The largest solution of $\ln(x+4) + \ln(x+5) = \ln(18x)$ is:

Solution: $\ln(x+4) + \ln(x+5) = \ln(18x)$ is equivalent to $\ln(x+4)(x+5) = \ln(18x)$, and since the ln function is 1:1, we must have $(x+4)(x+5) = 18x$ or $x^2 + 9x + 20 = 18x$ or $x^2 - 9x + 20 = 0$ or $(x-4)(x-5) =$. There are two solutions, 4 and 5, the largest of which is **5**

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(34) Find the absolute maximum value of the function $f(x) = 2 + x^2e^{2-x}$ over the interval $[0, 3]$.

Solution: $f'(x) = (x^2)'e^{2-x} + x^2\left(e^{2-x}\right)' = 2xe^{2-x} + x^2e^{2-x}(2-x)' = 2xe^{2-x} + x^2e^{2-x}(-1)' = e^{2-x}(2x - x^2) = 0$if $x = 0$ or 2. Thus the absolute maximum value occurs at 0, 2 or 3. We have

$f(0) = 2$, $f(2) = 2 + 2^2e^{2-2} = 6$, $f(3) = 2 + 3^2e^{2-3} = 2 + 9e^-1$. How can we compare this to 6 without a calculator? Easy: just look at $f'(3) = e^{2-3}(2(3) - 3^2) = e^{-1}(6-9) < 0$, so the function decreases from 2 to 3. Therefore the absolute maximum is **6**

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(35) Two non-negative numbers must add up to 16. If they are chosen so as to make their product as large as possible, then one-sixteenth of the maximum value of their product is:

Solution: Let x be one of the numbers, so that the other is $16 - x$. Let $f(x) = x(16 - x) = 16x - x^2$ be their product. Then $f'(x) = 16 - 2x = 0$ if $x = 8$, and $f''(x) = -2 < 0$, so a maximum occurs when $x = 8$. It is $f(8) = 64$, one-sixteenth of which is **4.**

Evaluate the logarithm:

If $\log_2 4096 = 10a + b$ then **(36)** $a =$ and **(37)** $b =$

Solution: Since $4096 = 2^{12}$, we have $\log_2 4096 = 12$ and thus

(36) $a =$ **1** and **(37)** $b =$ **2**

Evaluate the logarithm: If $\log_{125} 625 = \frac{c}{d}$ then **(38)** $c =$ and **(39)** $d =$

Solution: $\log_{125} 625 = \frac{\log_5 625}{\log_5 125} = \frac{\log_5 5^4}{\log_5 5^3} = \frac{4}{3}$, so

(38) $c =$ **4** and **(39)** $d =$ **3**

Evaluate the logarithm:If $\log_9 243 = \frac{e}{f}$ then **(40)** $e =$ and **(41)** $f =$

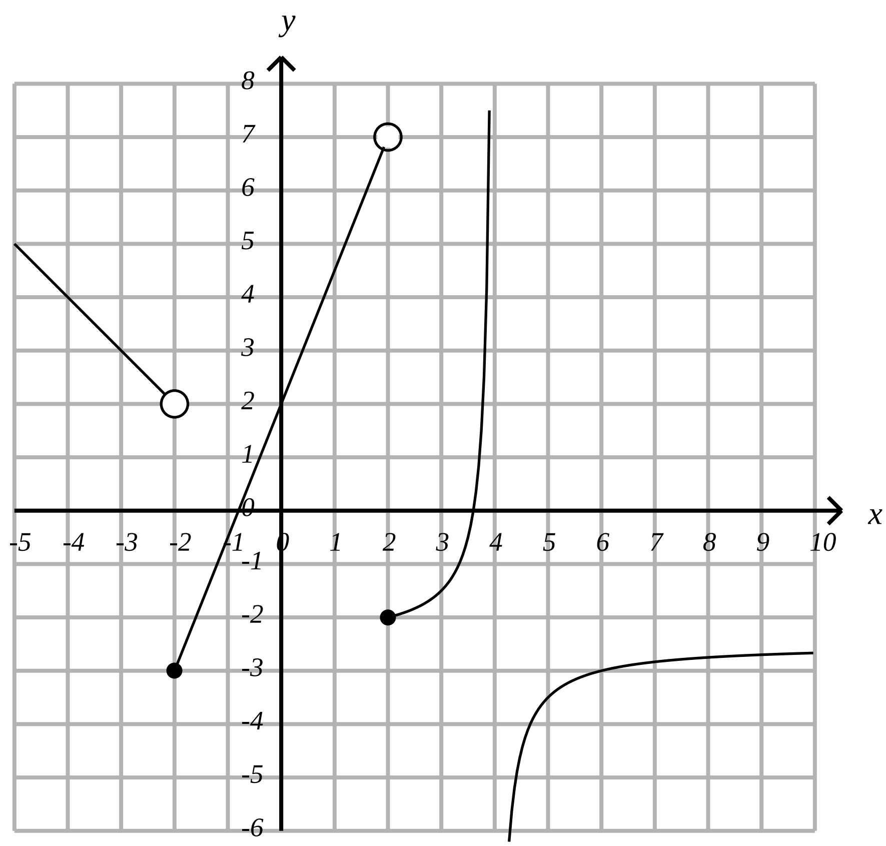
Solution: $\log_9 243 = \frac{\log_3 243}{\log_3 9} = \frac{\log_3 3^5}{\log_3 3^2} = \frac{5}{2}$, so **(40)** $e =$ **5** and **(41)** $f =$ **2**

PART II: The answers to all questions are the digits in **ANSWER SET II:**

(A) $-\infty$ (B) -3 (C) -2.5 (D) -2 (E) 0 (F) 1 (G) 2 (H) 4 (I) 7 (J) ∞

(42) $\lim_{x\to 5^-} \frac{6x-19}{x-5}$ =? **J**

(43) $\lim_{x\to 5^+} \frac{6x-19}{x-5}$ =? **A**



Part of the graph of $y = f(x) = \begin{cases} -x & \text{if } x < -2 \\ \frac{5}{2}x + 2 & \text{if } -2 \le x < 2 \\ \frac{1}{4-x} - 2.5 & \text{if } 2 \le x \text{ and } x \ne 4 \end{cases}$ is shown above.

Find:

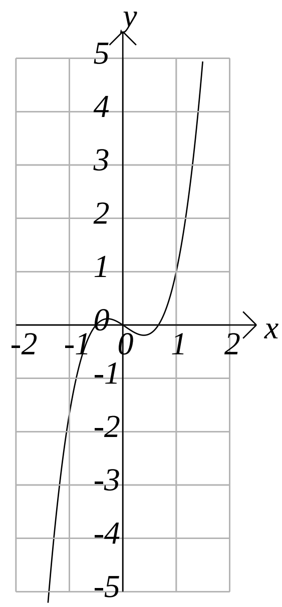
(44) $\lim_{x\to -\infty} f(x)$ **J** **(45)** $\lim_{x\to -2^-} f(x)$ **G** **(46)** $\lim_{x\to -2^+} f(x)$ **B** **(47)** $\lim_{x\to 2^-} f(x)$ **I**

(48) $\lim_{x\to 2^+} f(x)$ **D** **(49)** $\lim_{x\to 4^-} f(x)$ **J** **(50)** $\lim_{x\to 4^+} f(x)$ **A** **(51)** $\lim_{x\to \infty} f(x)$ **C**

…5

PART III

The graph of
$y = f(x) = x(2x+1)(3x-2)/3$
is shown to the right.
Parts of the graphs of



(52) $y = f(x+1)$, **D**

(53) $y = f(x) - 1$, **C**

(54) $y = f(x-1)$, **B**

(55) $y = -f(x)$, **A**

(56) $y = 2f(x)$, **J**

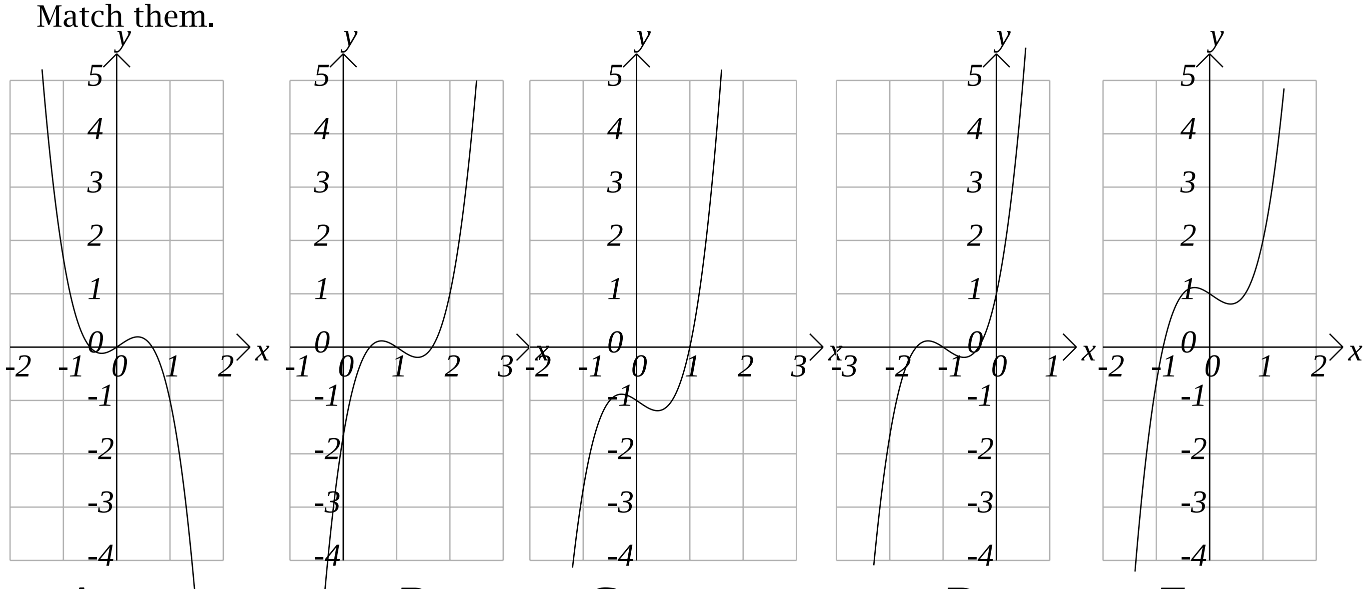
(57) $y = f(2x)$, **I**

(58) $y = f(x)/2$, **H**

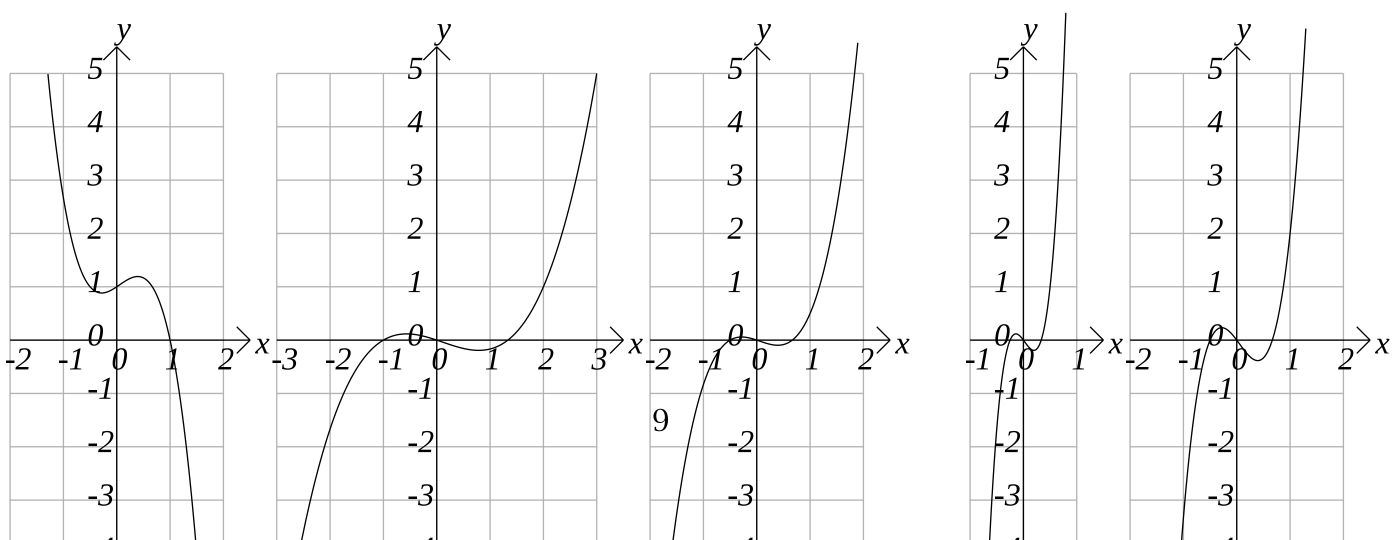
(59) $y = f(x/2)$, **G** and

(60) $y = -f(x) + 1$, **F** are shown below.

Match them.



A B C D E



The graphs of the functions in questions 61 to 70 are shown below. Match them.

(61) $f(x) = 4x^2 - 3x - 4$ **G**

(62) $f(x) = 2x^3 - x^4 + 1$ **F**

(63) $f(x) = 3x - 4x^2 - 1$ **C**

(64) $f(x) = \dfrac{12x + 3}{2x - 3}$ **I**

(65) $f(x) = \dfrac{1}{2}\left(3x - x^3\right)$ **B**

(66) $f(x) = \ln(\sqrt{5x + 3})$ **A**

(67) $f(x) = \dfrac{1}{10}\left(2x^3 - 3x^2 + 3\right)$ **D**

(68) $f(x) = xe^{-x+1}$ **H**

(69) $f(x) = \dfrac{1}{20}\left(x^4 - 4x^2\right) - 1$ **E**

(70) $f(x) = \dfrac{3x + 2}{3x - 2}$ **J**