# Sketching Rational Functions



Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \frac{p(x)}{q(x)}$. Things would be simpler if we could assume that $p$ and $q$ had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of $f$, and the roots of the denominator give us the possible locations of the vertical asymptotes of $f$. From the Quotient Rule $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of $f$, and the denominator of the second derivative is the fourth power of the denominator of $f$, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

---

**Example 1:** The graph of $f(x) = \frac{x+1}{x-1}$ is:

## Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$.

The only "interesting value" of $f$ is $-1$.

## Step 2:

Put these values of $x$ into increasing order. $-1.$

## Step 3:

Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of $f$.

| $x$ | $(-\infty, 1)$ | 1 | $(1, \infty)$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $UND$ | $+$ |
| $f'(x)$ | $-$ | $UND$ | $-$ |
| $f(x)$ | | $UND$ | |

**Step 4:** Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

But now we have a problem: the only interesting $x$-value, 1, doesn't have a corresponding $y$-value because 1 is not in the domain of $f$!

Even though the function is not defined at 1, we can compute the left- and right- hand limits there:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty$, and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty$.

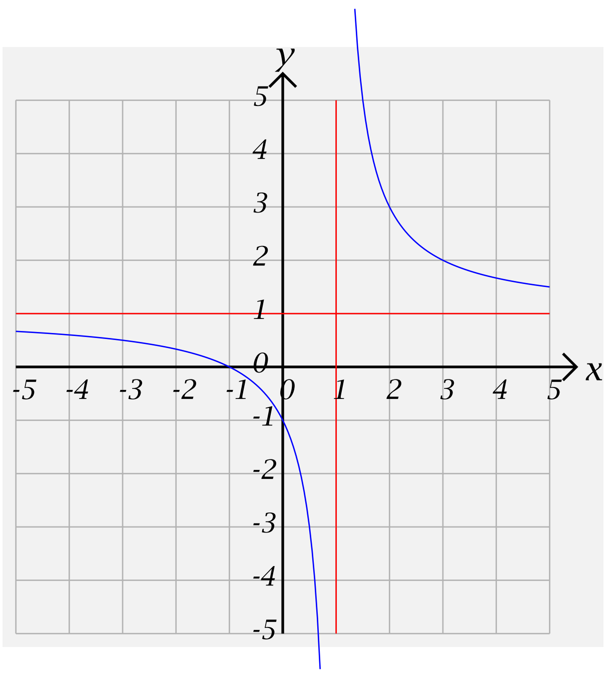
We can also compute the limits at $-\infty$ and $+\infty$:

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x+1}{x-1} = 1$, and $\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x+1}{x-1} = 1$.

We can also easily see that $f(-1) = 0$. We can represent these facts in a modified table, using a bit of shorthand notation:

| $x$ | $-\infty$ | $(-\infty, 1)$ | $1^-$ | $1^+$ | $(1, \infty)$ | $\infty$ |
|---|---|---|---|---|---|---|
| $f''(x)$ | 0 | $-$ | $-\infty$ | $+\infty$ | $+$ | 0 |
| $f'(x)$ | 0 | $-$ | $-\infty$ | $-\infty$ | $-$ | 0 |
| $f(x)$ | 1 | $f(-1) = 0$ | $-\infty$ | $\infty$ | | 1 |

We use this to put together a sketch of the graph: The horizontal line $y = 1$ and the vertical line $x = 1$ are called asymptotes of the graph.



**Example 2:** The graph of $f(x) = \dfrac{x}{x^2 - 1}$ is:

## Step 1:

$$f'(x) = \frac{(x^2-1)(x)' - (x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

and $f''(x) = -\dfrac{(x^2-1)^2(x^2+1)' - (x^2+1)\left((x^2-1)^2\right)'}{\left((x^2-1)^2\right)^2} =$

$$-\frac{(x^2-1)^2(2x) - (x^2+1)\left(2(x^2-1)(2x)\right)}{(x^2-1)^4} = -2x\frac{(x^2-1) - (x^2+1)(2)}{(x^2-1)^3} =$$

$$-2x\frac{x^2 - 1 - 2x^2 - 2}{(x^2-1)^3} = -2x\frac{-3 - x^2}{(x^2-1)^3} = 2x\frac{3 + x^2}{(x^2-1)^3}$$

The only "interesting values" of $f$ are $-1$, $0$, and $1$.

## Step 2:

Put these values of $x$ into increasing order. $-1, 0, 1.$

## Step 3:

Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of $f$.

| $x$ | $-\infty$ | $(-\infty,-1)$ | $-1^-$ | $-1^+$ | $(-1,0)$ | $0$ | $(0,1)$ | $1^-1$ | $1^+$ | $(1,\infty)$ | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f''(x)$ | $0$ | $-$ | $-\infty$ | $+\infty$ | $+$ | $0$ | $-$ | $-\infty$ | $\infty$ | $+$ | $0$ |
| $f'(x)$ | $0$ | $-$ | $-\infty$ | $-\infty$ | $-$ | $-1$ | $-$ | $-\infty$ | $-\infty$ | $-$ | $0$ |
| $f(x)$ | $0$ | $-$ | $-\infty$ | $\infty$ | $+$ | $0$ | $-$ | $-\infty$ | $\infty$ | $+$ | $0$ |

## Step 4:

Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

We use this to put together a sketch of the graph:

The horizontal line $y = 0$ and the vertical lines $x = -1$ and $x = 1$ are called asymptotes of the graph.