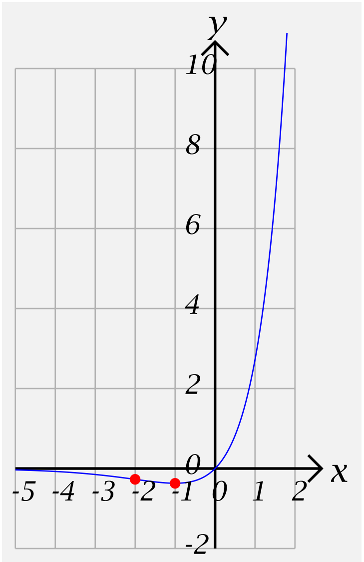
Sketching Products of Log and Exponential Functions

Example 1: $f(x) = xe^x$

Step 1: $f'(x) = (x)'e^x + x(e^x)' = (1)e^x + xe^x = (x+1)e^x$ and $f''(x) = (x+1)'e^x + (x+1)(e^x)' = (1)e^x + (x+1)e^x = (x+2)e^x$. The "interesting values" of f are -1 and -2, which divide the domain $(-\infty, \infty)$ into three intervals: $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.

Step 2: Put these values of x into increasing order. $-2, -1$



Step 3: Put together as good a table as you can.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, \infty)$
$f''(x)$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$-$	0	$+$
$f(x)$		$-\frac{2}{e^2} \doteq -0.27$		$-\frac{1}{e} \doteq -0.36$	$f(0) = 0$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

Example 2: $f(x) = xe^{-2x}$

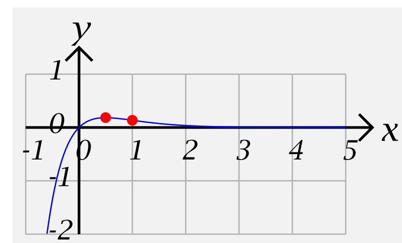
Step 1: $f'(x) = (x)'e^{-2x} + x\left(e^{-2x}\right)' = (1)e^{-2x} + xe^{-2x}(-2) = (-2x+1)e^{-2x}$ and $f''(x) = (-2x+1)'e^{-2x} + (-2x+1)\left(e^{-2x}\right)' = (-2)e^{-2x} + (-2x+1)e^{-2x}(-2) = (4x-4)e^{-2x} = 4(x-1)e^{-2x}$. The "interesting values" of f are $\frac{1}{2}$ and 1, which divide the domain $(-\infty, \infty)$ into three intervals: $\left(-\infty, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 1\right)$ and $(1, \infty)$.

Step 2: Put these values of x into increasing order. $\frac{1}{2}, 1$

Step 3: Put together as good a table as you can.

x	$\left(-\infty, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$	1	$(1, \infty)$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f(x)$	$f(0) = 0$	$\frac{1}{2e} \doteq 0.183$		$\frac{1}{e^2} \doteq 0.135$	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 3: $f(x) = x^2 e^{-2x}$

Step 1: $f'(x) = (x^2)'e^{-2x} + x^2\left(e^{-2x}\right)' = (2x)e^{-2x} + x^2 e^{-2x}(-2) = (-2x^2+2x)e^{-2x} = -2x(x-1)$ and $f''(x) = (-2x^2+2x)'e^{-2x} + (-2x^2+2x)\left(e^{-2x}\right)' = (-4x+2)e^{-2x} + (-2x^2+2x)e^{-2x}(-2) = (-4x+2)e^{-2x} + (4x^2-4x)e^{-2x} = (4x^2-8x+2)e^{-2x} = 2(2x^2-4x+1)e^{-2x}$.

The "interesting values" from f' are 0 and 1, and those of f'' are:

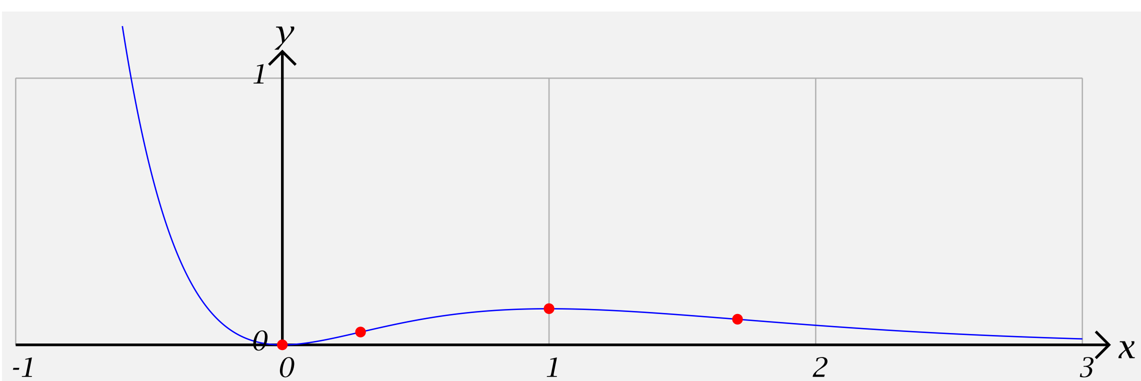
$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)1}}{2(2)} = \frac{4 \pm \sqrt{16-8}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2} = 1 \pm \frac{\sqrt{2}}{2}.$$

Step 2: Put these values of x into increasing order. $0, 1-\frac{\sqrt{2}}{2}, 1, 1+\frac{\sqrt{2}}{2}$

Step 3: Put together as good a table as you can.

x	$(-\infty, 0)$	0	$\left(0, 1-\frac{\sqrt{2}}{2}\right)$	$1-\frac{\sqrt{2}}{2}$	$\left(1-\frac{\sqrt{2}}{2}, 1\right)$	1	$\left(1, 1+\frac{\sqrt{2}}{2}\right)$	$1+\frac{\sqrt{2}}{2}$	$\left(1+\frac{\sqrt{2}}{2}, \infty\right)$
$f''(x)$	+	+	+	0	−	−	−	0	+
$f'(x)$	−	0	+	+	+	0	−	−	−
$f(x)$	+	0	+	+	+	$\frac{1}{e^2} \doteq 0.135$	+	+	+

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 4: $f(x) = x^2 e^{2x}$

Step 1: $f'(x) = (x^2)'e^{2x} + x^2\left(e^{2x}\right)' = (2x)e^{2x} + x^2e^{2x}(2) = (2x^2+2x)e^{2x} = 2x(x+1)e^{2x}$ and $f''(x) = (2x^2+2x)'e^{2x} + (2x^2+2x)\left(e^{2x}\right)' = (4x+2)e^{2x} + (2x^2+2x)e^{2x}(2) = (4x+2)e^{-2x} + (4x^2+4x)e^{2x} = (4x^2+8x+2)e^{2x} = 2(2x^2+4x+1)e^{2x}$.

The “interesting values” from f' are 0 and 1, and those of f'' are:

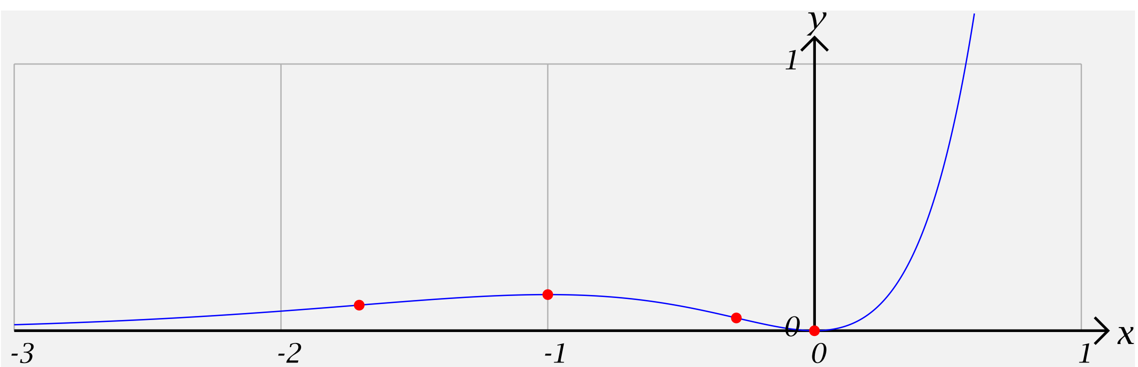
$$\frac{-(4) \pm \sqrt{(4)^2 - 4(2)1}}{2(2)} = \frac{-4 \pm \sqrt{16-8}}{4} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2} = -1 \pm \frac{\sqrt{2}}{2}.$$

Step 2: Put these values of x into increasing order. $-1-\frac{\sqrt{2}}{2}, -1, -1+\frac{\sqrt{2}}{2}, 0$

Step 3: Put together as good a table as you can.

x	$\left(-\infty, \frac{-2-\sqrt{2}}{2}\right)$	$-1-\frac{\sqrt{2}}{2}$	$\left(-1-\frac{\sqrt{2}}{2}, -1\right)$	-1	$\left(-1, -1+\frac{\sqrt{2}}{2}\right)$	$-1+\frac{\sqrt{2}}{2}$	$\left(-1+\frac{\sqrt{2}}{2}, 0\right)$	0	$(0,\infty)$
$f''(x)$	+	0	−	−	−	0	+	+	+
$f'(x)$	+	+	+	0	−	−	−	0	+
$f(x)$	+	0	+	$\frac{1}{e^2}$	+	+	+	0	+

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 5: $f(x) = xe^{x^2}$

Step 1: $f'(x) = (x)'e^{x^2} + x\left(e^{x^2}\right)' = (1)e^{x^2} + xe^{x^2}(2x) = (2x^2+1)e^{x^2} > 0$ and $f''(x) = (2x^2+1)'e^{x^2} + (2x^2+1)\left(e^{x^2}\right)' = (4x)e^{x^2} + (2x^2+1)e^{x^2}(2x) = (4x)e^{x^2} + (4x^3+2x)e^{x^2} = (4x^3+6x)e^{x^2} = 2x(2x^2+3)e^{x^2}$.

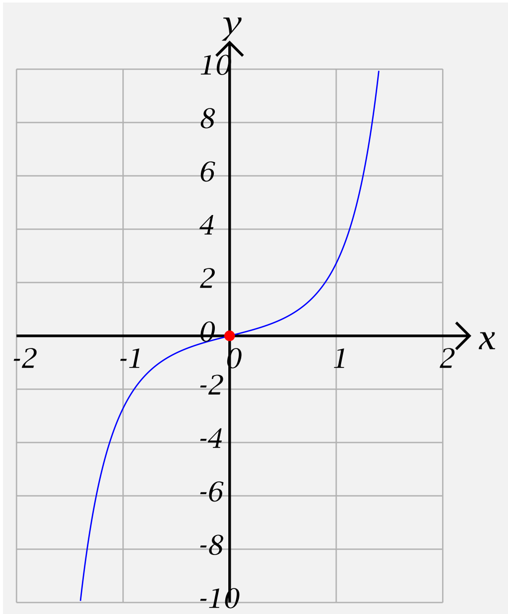
The only "interesting values" is 0.

Step 2: Put these values of x into increasing order. 0

Step 3: Put together as good a table as you can.

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	0	$-+$
$f'(x)$	$+$	$+$	$+-$
$f(x)$	$-$	0	$+$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 6: $f(x) = e^{-\frac{1}{x^2}}$

Step 1: $f'(x) = e^{-\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)' = e^{-\frac{1}{x^2}}\left(-(-2)x^{-3}\right) = 2x^{-3}e^{-\frac{1}{x^2}}$ and

$$f''(x) = (2x^{-3})'\,e^{-\frac{1}{x^2}} + (2x^{-3})\left(e^{-\frac{1}{x^2}}\right)' = \left(2(-3)x^{-4}\right)e^{-\frac{1}{x^2}} + (2x^{-3})\,e^{-\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)' =.$$

$$(-6x^{-4})\,e^{-\frac{1}{x^2}} + (2x^{-3})\,e^{-\frac{1}{x^2}}\,(2x^{-3}) = \left(\frac{-6}{x^4}+\frac{4}{x^6}\right)e^{-\frac{1}{x^2}} = \left(\frac{-6x^2+4}{x^6}\right)e^{-\frac{1}{x^2}} = -6x^{-6}\left(x^2-\frac{2}{3}\right)e^{-\frac{1}{x^2}}.$$

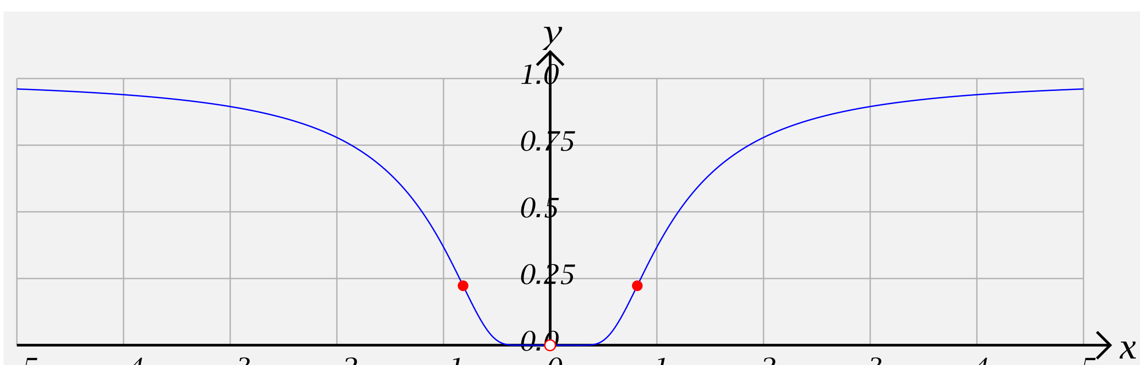
The “interesting values” 0 and $\pm\sqrt{\frac{2}{3}}$.

Step 2: Put these values of x into increasing order. $-\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}}$

Step 3: Put together as good a table as you can.

x	$\left(-\infty, -\sqrt{\frac{2}{3}}\right)$	$-\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, 0\right)$	0	$\left(0, \sqrt{\frac{2}{3}}\right)$	$\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, \infty\right)$
$f''(x)$	$+$	0	$-$	*und*	$-$	0	$+$
$f'(x)$	$-$	$-$	$-$	*und*	$+$	$+$	$+$
$f(x)$	≤ 1	$e^{-\frac{3}{2}} \doteq 0.22$	$+$	*und*	$+$	$e^{-\frac{3}{2}} \doteq 0.22$	≤ 1

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 7: $f(x) = x \ln x$

Step 1: The domain of f is $(0, \infty)$. $f'(x) = (x)' \ln x + x(\ln x)' = (1)\ln x + x\frac{1}{x} = \ln x + 1 = 0$ if $\ln x = -1$. Taking exponentials, we get $x = e^{\ln x} = e^{-1} = \frac{1}{e}$

Also $f''(x) = \frac{1}{x}$.

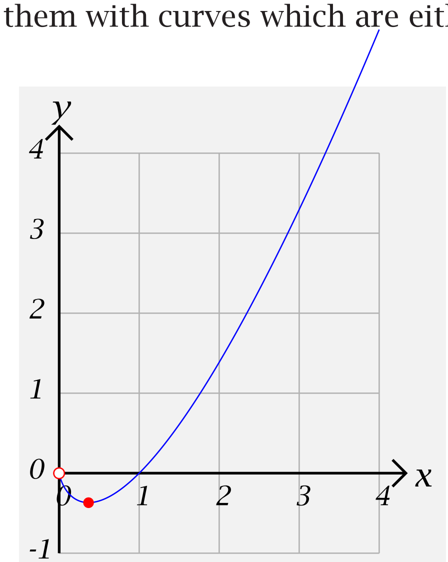
The "interesting values" are 0 and $\frac{1}{e}$.

Step 2: Put these values of x into increasing order. $0, \frac{1}{e}$

Step 3: Put together as good a table as you can.

x	$\left(0, \frac{1}{e}\right)$	$\frac{1}{e}$	$\left(\frac{1}{e}, \infty\right)$
$f''(x)$	+	+	+
$f'(x)$	−	0	+
$f(x)$	−	$-\frac{1}{e} \doteq -0.37$	$f(1) = 0$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 8: $f(x) = x^2 \ln x$

Step 1: The domain of f is $(0, \infty)$. $f'(x) = (x^2)' \ln x + (x^2)(\ln x)' = 2x \ln x + x^2 \dfrac{1}{x} = 2x \ln x + x = x(2 \ln x + 1) = 0$ if $x = 0$ or $\ln x = -\dfrac{1}{2}$. Taking exponentials, we get $x = e^{\ln x} = e^{-\frac{1}{2}} = \dfrac{1}{\sqrt{e}}$

Also $f''(x) = (2x)' \ln x + (2x)(\ln x)' + 1 = 2 \ln x + 2x\dfrac{1}{x} + 1 = 2 \ln x + 3 = 0$ if $\ln x = -\dfrac{3}{2}$.
Taking exponentials, we get $x = e^{\ln x} = e^{-\frac{3}{2}} = \dfrac{1}{e\sqrt{e}}$.

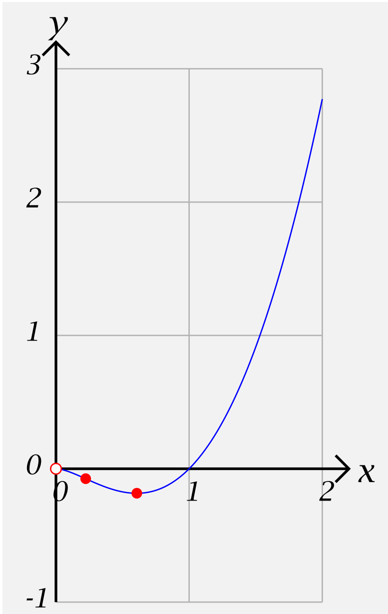
The "interesting values" are 0, $\dfrac{1}{\sqrt{e}}$, and $\dfrac{1}{e\sqrt{e}}$.

Step 2: Put these values of x into increasing order. $0, \dfrac{1}{e\sqrt{e}}, \dfrac{1}{\sqrt{e}}$

Step 3: Put together as good a table as you can.

x	$\left(0, \frac{1}{e\sqrt{e}}\right)$	$\frac{1}{e\sqrt{e}}$	$\left(\frac{1}{e\sqrt{e}}, \frac{1}{\sqrt{e}}\right)$	$\frac{1}{\sqrt{e}}$	$\left(\frac{1}{\sqrt{e}}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	+
$f'(x)$	+	0	$-$	$-$	$-$
$f(x)$	$-$	$-\frac{3}{2e^3} \doteq -0.07$		$-\frac{1}{2e} \doteq -0.18$	$f(1) = 0$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 9: $f(x) = \dfrac{\ln x}{x^2}$

Step 1: The domain of f is $(0, \infty)$.

$f'(x) = \dfrac{(x^2)(\ln x)' - (\ln x)(x^2)'}{(x^2)^2} = \dfrac{(x^2)\left(\frac{1}{x}\right) - (\ln x)(2x)}{x^4} = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3} = 0$ if $\ln x = \dfrac{1}{2}$. Taking exponentials, we get $x = e^{\ln x} = e^{\frac{1}{2}} = \sqrt{e}$

Also $f''(x) = \dfrac{(x^3)(1 - 2\ln x)' - (1 - 2\ln x)(x^3)'}{(x^3)^2} = \dfrac{(x^3)\left(-2\frac{1}{x}\right) - (1 - 2\ln x)(3x^2)}{x^6} = \dfrac{-2x^2 - (1 - 2\ln x)(3x^2)}{x^6} = \dfrac{-2 - (1 - 2\ln x)(3)}{x^4} =$
$\dfrac{6\ln x - 5}{x^4} = 0$ if $\ln x = \dfrac{5}{6}$.
Taking exponentials, we get $x = e^{\ln x} = e^{\frac{5}{6}}$.

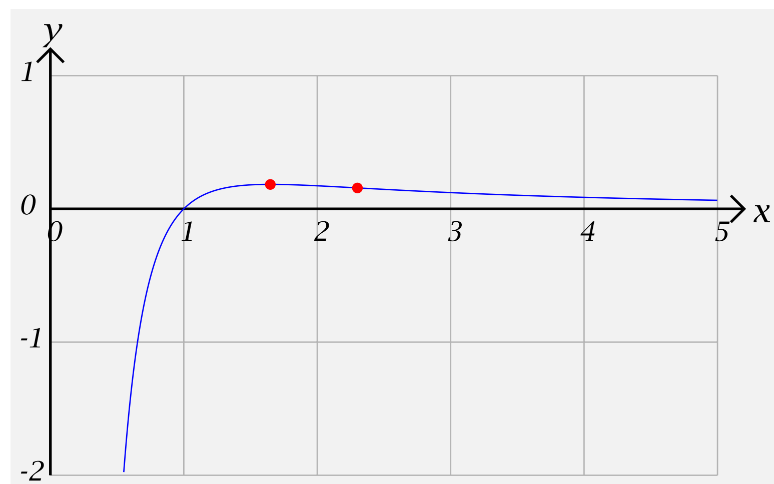
The "interesting values" are 0, $e^{\frac{1}{2}}$, and $e^{\frac{5}{6}}$.

Step 2: Put these values of x into increasing order. $0, e^{\frac{1}{2}}, e^{\frac{5}{6}}$

Step 3: Put together as good a table as you can.

x	$\left(0, e^{\frac{1}{2}}\right)$	$e^{\frac{1}{2}}$	$\left(e^{\frac{1}{2}}, e^{\frac{5}{6}}\right)$	$e^{\frac{5}{6}}$	$\left(e^{\frac{5}{6}}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f(x)$	$-$	$\dfrac{1}{2e} \doteq 1.36$		$\dfrac{5}{6e^{\frac{5}{3}}} \doteq 0.16$	$f(1) = 0$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 10: $f(x) = \dfrac{\ln x}{x^{\frac{1}{2}}}$

Step 1: The domain of f is $(0, \infty)$.

$$f'(x) = \frac{(x^{\frac{1}{2}})(\ln x)' - (\ln x)(x^{\frac{1}{2}})'}{(x^{\frac{1}{2}})^2} = \frac{(x^{\frac{1}{2}})\left(\frac{1}{x}\right) - (\ln x)(\frac{1}{2}x^{-\frac{1}{2}})}{x} = \frac{\frac{1}{x^{\frac{1}{2}}} - \frac{\frac{1}{2}\ln x}{x^{\frac{1}{2}}}}{x} =$$

$$\frac{2 - \ln x}{2x^{\frac{3}{2}}} = 0 \text{if } \ln x = 2 \text{ or } x = e^2$$

$$\text{Also } f''(x) = \frac{\left(2x^{\frac{3}{2}}\right)(2 - \ln x)' - (2 - \ln x)\left(2x^{\frac{3}{2}}\right)'}{\left(2x^{\frac{3}{2}}\right)^2} = \frac{\left(2x^{\frac{3}{2}}\right)\left(-\frac{1}{x}\right) - (2 - \ln x)\left(2\frac{3}{2}x^{\frac{1}{2}}\right)}{4x^3} =$$

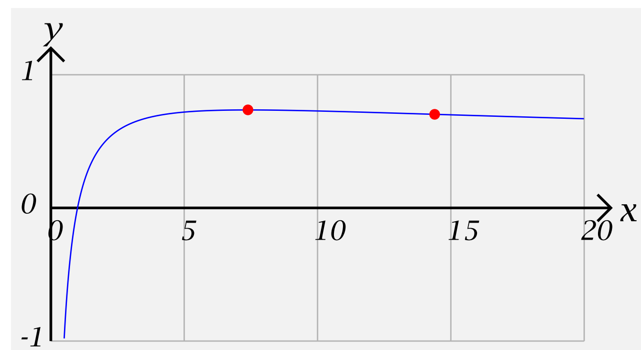
$$\frac{-2x^{\frac{1}{2}} - (2 - \ln x)\left(3x^{\frac{1}{2}}\right)}{4x^3} = \frac{-2 - 3(2 - \ln x)}{4x^{\frac{5}{2}}} = \frac{3\ln x - 8}{4x^{\frac{5}{2}}} = 0 \text{ if } \ln x = \frac{8}{3} \text{ or } x = e^{\frac{8}{3}}$$

The "interesting values" are 0, e^2, and $e^{\frac{8}{3}}$.

Step 2: Put these values of x into increasing order. $0, e^2, e^{\frac{8}{3}}$

Step 3: Put together as good a table as you can.

x	$(0, e^2)$	e^2	$\left(e^2, e^{\frac{8}{3}}\right)$	$e^{\frac{8}{3}}$	$\left(e^{\frac{8}{3}}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f(x)$	$f(1) = 0$	$\frac{2}{e} \doteq 0.74$		$\frac{8}{3e^{\frac{4}{3}}} \doteq 0.70$	



Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.