Sketching Functions with Rational Powers

When dealing with functions containing fractional powers special attention must be given to determining the domain of the function. It is crucial to detect even denominators in the fractional powers, because these will cause the functions to be undefined wherever an even root of a negative number would occur. It is also important to be able to simplify the first and second derivatives by factoring.

If all of the powers are positive, the function is said to be algebraic, and will have no vertical asymptotes, although vertical tangent lines are possible. If there are negative powers, vertical asymptotes may be present.

Example 1: Sketch the graph of $f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$.

Step 1: The domain is $(-\infty, \infty)$.

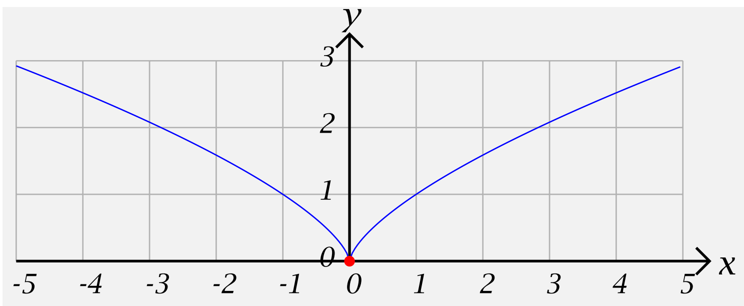
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, and $f''(x) = -\frac{2}{9}x^{-\frac{4}{3}}$, so the only "interesting value" of f is 0.

Step 2: Put these values of x into increasing order. 0.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	UND	$-$
$f'(x)$	$-$	UND	$+$
$f(x)$	$+$	0	$+$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 2: $f(x) = \sqrt{x^2 - 1} = (x^2 - 1)^{\frac{1}{2}}$

Step 1: The domain of f is $(-\infty, 0] \cup [1, \infty)$.

$$f'(x) = \frac{1}{2}(x^2-1)^{-\frac{1}{2}}(x^2-1)' = \frac{x}{(x^2-1)^{\frac{1}{2}}} \text{ and}$$

$$f''(x) = \frac{(x^2-1)^{\frac{1}{2}}(x)' - x\left((x^2-1)^{\frac{1}{2}}\right)'}{\left((x^2-1)^{\frac{1}{2}}\right)^2} = \frac{(x^2-1)^{\frac{1}{2}}(1) - x\left(\frac{x}{(x^2-1)^{\frac{1}{2}}}\right)}{|x^2-1|} =$$

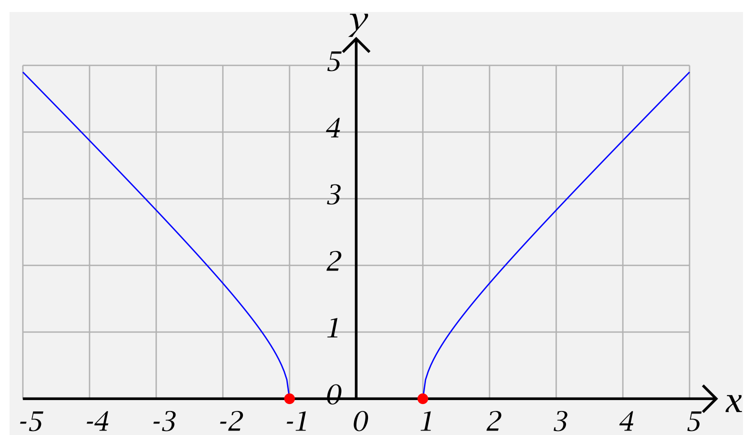
$$\frac{(x^2-1)^{\frac{1}{2}} - \frac{x^2}{(x^2-1)^{\frac{1}{2}}}}{|x^2-1|} = \frac{\frac{x^2-1}{(x^2-1)^{\frac{1}{2}}} - \frac{x^2}{(x^2-1)^{\frac{1}{2}}}}{|x^2-1|} = \frac{\frac{-1}{(x^2-1)^{\frac{1}{2}}}}{|x^2-1|} = \frac{-1}{(x^2-1)^{\frac{3}{2}}}$$

The "interesting values" of f are -1 and 1. (Remember, 0 is not in the domain of f).

Step 2: Put these values of x into increasing order. $-1, 1$

Step 3: Put together as good a table as you can.

x	$(-\infty, -1)$	-1	1	$(1, \infty)$
$f''(x)$	$-$	$-\infty$	$-\infty$	$-$
$f'(x)$	$-$	$-\infty$	∞	$+$
$f(x)$	$+$	0	0	$+$



Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

Example 3: Sketch the graph of $f(x) = \sqrt[3]{x^2-4} = (x^2-4)^{\frac{1}{3}}$.

Step 1: The domain of f is $(-\infty, \infty)$.

$f'(x) = \frac{1}{3}(x^2-4)^{-\frac{2}{3}}(x^2-4)' = \frac{2}{3}\frac{x}{(x^2-4)^{\frac{2}{3}}}$ and

$$f''(x) = \frac{2}{3}\frac{(x^2-4)^{\frac{2}{3}}(x)' - x\left((x^2-4)^{\frac{2}{3}}\right)'}{\left((x^2-4)^{\frac{2}{3}}\right)^2} = \frac{2}{3}\frac{(x^2-4)^{\frac{2}{3}}(1) - x\left(\frac{2}{3}\frac{2x}{(x^2-4)^{\frac{1}{3}}}\right)}{(x^2-4)^{\frac{4}{3}}} =$$

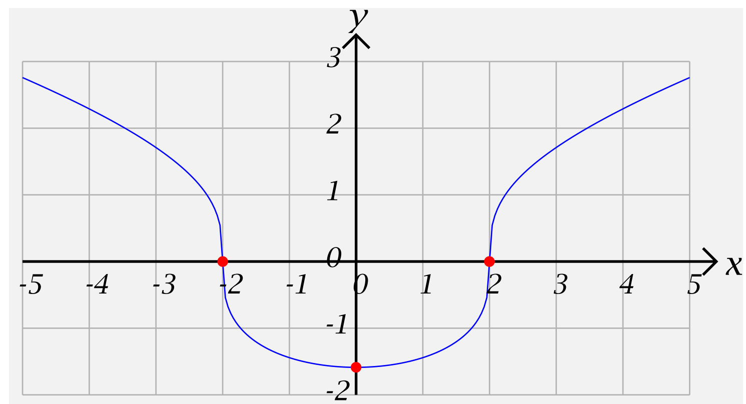
$$\frac{2}{3}\frac{(x^2-4)^{\frac{2}{3}}\frac{(x^2-4)^{\frac{1}{3}}}{(x^2-4)^{\frac{1}{3}}} - \left(\frac{4}{3}\frac{x^2}{(x^2-4)^{\frac{1}{3}}}\right)}{(x^2-4)^{\frac{4}{3}}} = \frac{2}{3}\frac{(x^2-4)^1 - \frac{4}{3}x^2}{(x^2-4)^{\frac{5}{3}}} = \frac{2}{3}\left(\frac{-4-\frac{1}{3}x^2}{(x^2-4)^{\frac{5}{3}}}\right) = -\frac{2}{9}\left(\frac{12+x^2}{(x^2-4)^{\frac{5}{3}}}\right)$$

The "interesting values" of f are -2, 0, and 2.

Step 2: Put these values of x into increasing order. $-2, 0, 2$

Step 3: Put together as good a table as you can.

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,2)$	2	$(2,\infty)$
$f''(x)$	$-$	UND	+	+	+	UND	$-$
$f'(x)$	$-$	UND	$-$	0	+	UND	+
$f(x)$	+	0	$-$	$-$	$-$	0	+



Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

Example 4: Sketch $f(x) = 2x + 3x^{\frac{2}{3}}$

Step 1: $f'(x) = 2 + 2x^{-\frac{1}{3}}$ and $f''(x) = -\frac{2}{3}x^{-\frac{4}{3}}$.

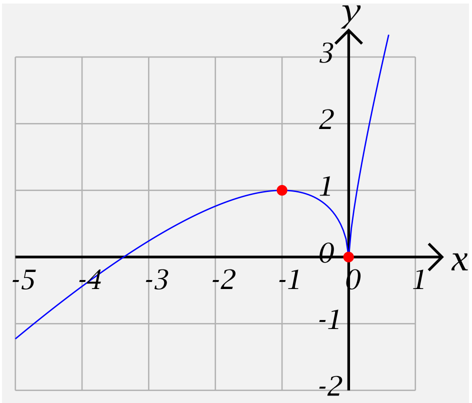
The "interesting values" of f' are -1 and 0, and the interesting value of f'' is 0.

Step 2: Put these values of x into increasing order. $-1, 0.$

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	$-$	$-$	UND	$-$
$f'(x)$	$+$	0	$-$	UND	$+$
$f(x)$		1		0	$+$

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 5: Sketch $f(x) = 4x - 3x^{\frac{4}{3}}$

Step 1: $f'(x) = 4 - 4x^{\frac{1}{3}}$ and $f''(x) = -\frac{4}{3}x^{-\frac{2}{3}}$.

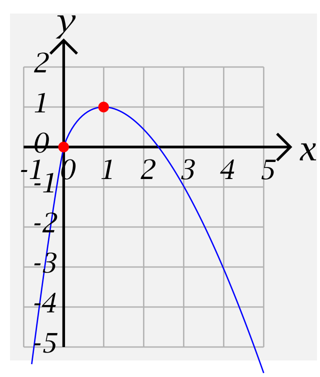
The “interesting value” of f' is 1, and the interesting value of f'' is 0.

Step 2: Put these values of x into increasing order. $0, 1.$

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0^-	0^+	$(0,1)$	1	$(1,\infty)$
$f''(x)$	$-$	$-\infty$	$-\infty$	$-$	$-$	$-$
$f'(x)$	$+$	4	4	$+$	0	$-$
$f(x)$	$-$	0	0	$+$	1	$-$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 6: $f(x) = x\sqrt{3-x} = x(3-x)^{\frac{1}{2}}$

Step 1: The domain of f is $(-\infty, 3]$.

$$f'(x) = (x)'(3-x)^{\frac{1}{2}} + x\left((3-x)^{\frac{1}{2}}\right)' = (1)(3-x)^{\frac{1}{2}} + x\left(\tfrac{1}{2}(3-x)^{-\frac{1}{2}}(3-x)'\right) =$$

$$(3-x)^{\frac{1}{2}} + x\left(\frac{1}{2}(3-x)^{-\frac{1}{2}}(-1)\right) = (3-x)^{\frac{1}{2}} - \frac{1}{2}x(3-x)^{-\frac{1}{2}} =$$

$$\left[3 - x - \frac{1}{2}x\right](3-x)^{-\frac{1}{2}} = \left[3 - \frac{3}{2}x\right](3-x)^{-\frac{1}{2}} = \frac{3}{2}\frac{2-x}{(3-x)^{\frac{1}{2}}}$$

and $f''(x) = \dfrac{3}{2}\dfrac{(3-x)^{\frac{1}{2}}(2-x)' - (2-x)\left((3-x)^{\frac{1}{2}}\right)'}{\left((3-x)^{\frac{1}{2}}\right)^2} =$

$$\frac{3}{2}\frac{(3-x)^{\frac{1}{2}}(-1) - (2-x)\left(\frac{1}{2}(3-x)^{-\frac{1}{2}}(3-x)'\right)}{|3-x|} =$$

$$\frac{3}{2}\frac{-(3-x)^{\frac{1}{2}} - (2-x)\left(\frac{1}{2}(3-x)^{-\frac{1}{2}}(-1)\right)}{|3-x|} =$$

$$\frac{3}{2}\frac{-(3-x) - (2-x)\left(\frac{1}{2}(-1)\right)}{(3-x)^{\frac{1}{2}}|3-x|} = \frac{3}{2}\frac{x - 3 + (2-x)\frac{1}{2}}{(3-x)^{\frac{3}{2}}} = \frac{3}{2}\frac{\frac{x}{2} - 2}{(3-x)^{\frac{3}{2}}} = \frac{3}{4}\frac{x-4}{(3-x)^{\frac{3}{2}}} =$$

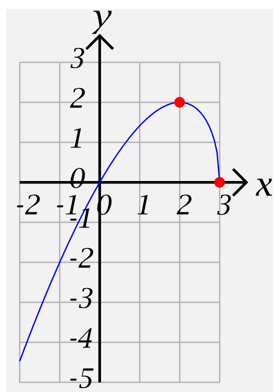
The "interesting values" of f are 2 and 3.

Step 2: Put these values of x into increasing order. 2, 3.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 2)$	2	$(2, 3)$	3
$f''(x)$	$-$	$-$	$-$	$-\infty$
$f'(x)$	$+$	0	$-$	$-\infty$
$f(x)$	$-$	2		0

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 7: Sketch the graph of $f(x) = 4x^{\frac{1}{3}} - x^{\frac{4}{3}}$

Step 1:

$f'(x) = \frac{4}{3}x^{-\frac{2}{3}} - \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}x^{-\frac{2}{3}}(1-x)$

and $f''(x) = -\dfrac{2}{9}x^{-\frac{5}{3}} - \dfrac{4}{9}x^{-\frac{2}{3}} = -\dfrac{2}{9}x^{-\frac{5}{3}}(1+2x)$

The "interesting values" of f are $-\dfrac{1}{2}$, 0, and 1.

Step 2:

Put these values of x into increasing order. $-\dfrac{1}{2}, 0, 1.$

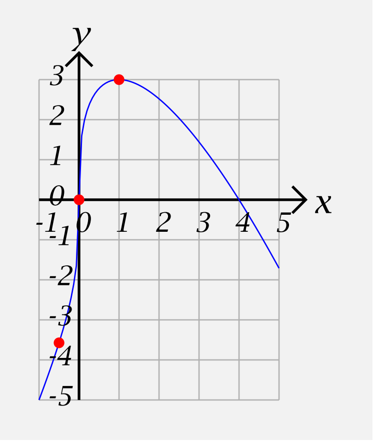
Step 3:

Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$\left(-\infty, -\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2}, 0\right)$	0	$(0,1)$	1	$(1,\infty)$
$f''(x)$	$-$	0	$+$	*UND*	$-$	$-$	$-$
$f'(x)$	$+$	$+$	$+$	*UND*	$+$	0	$-$
$f(x)$	$-$	$-$	$-$	0	$+$	3	

Step 4:

Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 8: $f(x) = \dfrac{8(x^{\frac{1}{2}} - 1)}{x}$

Step 1: The domain of f is $(0, \infty)$. $f(x) = 8x^{-\frac{1}{2}} - 8x^{-1}$, so

$$f'(x) = -8\tfrac{1}{2}x^{-\frac{3}{2}} + 8x^{-2} = -4x^{-\frac{3}{2}} + 8x^{-2} = 4x^{-2}\left(-x^{\frac{1}{2}} + 2\right)$$

and $f''(x) = (-4)\left(-\dfrac{3}{2}\right)x^{-\frac{5}{2}} - 16x^{-3} = 6x^{-\frac{5}{2}} - 16x^{-3} = 2x^{-3}\left(3x^{\frac{1}{2}} - 8\right)$

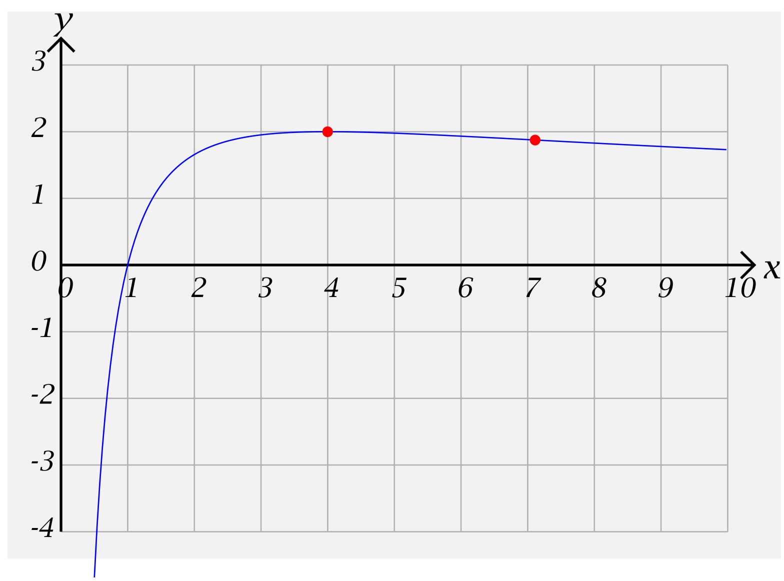
The "interesting values" of f are $0, 4$ and $\dfrac{64}{9}$.

Step 2: Put these values of x into increasing order. $0, 4, \dfrac{64}{9}$.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(0,4)$	4	$\left(4, \frac{64}{9}\right)$	$\frac{64}{9}$	$\left(\frac{64}{9}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f(x)$	$f(1) = 0$	2		$\frac{15}{8}$	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 9: $f(x) = \dfrac{1+x^{\frac{1}{2}}}{1-x^{\frac{1}{2}}}$

Step 1: The domain of f is $(0,1) \cup (,\infty)$.

$$f'(x) = \frac{\left(1-x^{\frac{1}{2}}\right)\left(1+x^{\frac{1}{2}}\right)' - \left(1+x^{\frac{1}{2}}\right)\left(1-x^{\frac{1}{2}}\right)'}{\left(1-x^{\frac{1}{2}}\right)^2} = \frac{\left(1-x^{\frac{1}{2}}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - \left(1+x^{\frac{1}{2}}\right)\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)}{\left(1-x^{\frac{1}{2}}\right)^2} =$$

$$\frac{1}{2}x^{-\frac{1}{2}}\frac{\left(1-x^{\frac{1}{2}}\right)+\left(1+x^{\frac{1}{2}}\right)}{\left(1-x^{\frac{1}{2}}\right)^2} = \frac{1}{x^{\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2} = \left(x^{\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2\right)^{-1}$$

and $f''(x) = (-1)\left(x^{\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2\right)^{-2}\left(x^{\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2\right)' =$

$$\frac{-1}{x\left(1-x^{\frac{1}{2}}\right)^4}\left[\left(x^{\frac{1}{2}}\right)'\left(1-x^{\frac{1}{2}}\right)^2 + x^{\frac{1}{2}}\left(\left(1-x^{\frac{1}{2}}\right)^2\right)'\right] =$$

$$\frac{-1}{x\left(1-x^{\frac{1}{2}}\right)^4}\left[\frac{1}{2}x^{-\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2 + x^{\frac{1}{2}}\left(2\left(1-x^{\frac{1}{2}}\right)\frac{1}{2}x^{-\frac{1}{2}}\right)\right] =$$

$$\frac{-1}{x\left(1-x^{\frac{1}{2}}\right)^3}\left[\frac{1}{2}x^{-\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)+1\right] = \frac{-1}{x\left(1-x^{\frac{1}{2}}\right)^3}\left[\frac{1}{2\sqrt{x}}+\frac{1}{2}\right]$$

so the only “interesting value” of f is 1.

Step 2: Put these values of x into increasing order. 1

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(0,1)$	1	$(1,\infty)$
$f''(x)$	$+$	UND	$-$
$f'(x)$	$+$	UND	$+$
$f(x)$	$+$	UND	$-$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.