

Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \frac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f , and the roots of the denominator give us the possible locations of the vertical asymptotes of f . From the Quotient Rule $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f , and the denominator of the second derivative is the fourth power of the denominator of f , so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \frac{x+1}{x-1}$ is:

Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

$$\text{and } f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}.$$

The only “interesting value” of f is -1 .

Step 2: Put these values of x into increasing order. -1 .

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f .

x	$(-\infty, 1)$	1	$(1, \infty)$
$f''(x)$	$-$	<i>UND</i>	$+$
$f'(x)$	$-$	<i>UND</i>	$-$
$f(x)$		<i>UND</i>	

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.

But now we have a problem: the only interesting x -value, 1, doesn't have a corresponding y -value because 1 is not in the domain of f !

Even though the function is not defined at 1, we can compute the left- and right- hand limits there:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty, \text{ and } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty.$$

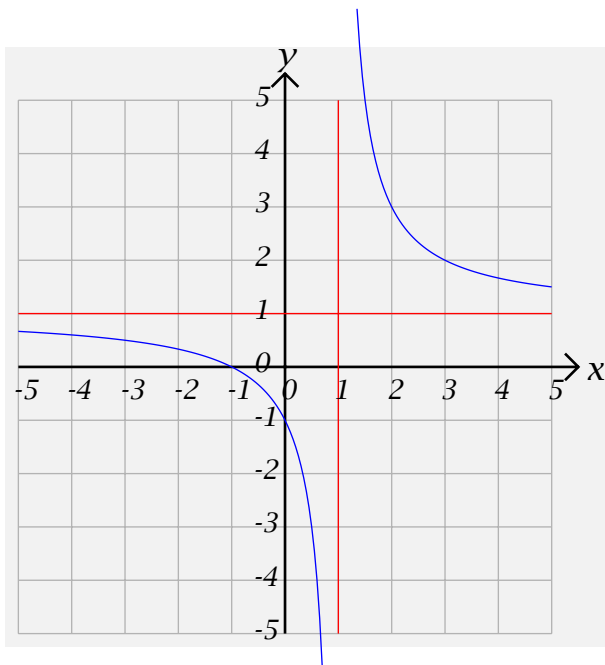
We can also compute the limits at $-\infty$ and $+\infty$:

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x+1}{x-1} = 1, \text{ and } \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x+1}{x-1} = 1.$$

We can also easily see that $f(-1) = 0$. We can represent these facts in a modified table, using a bit of shorthand notation:

x	$-\infty$	$(-\infty, 1)$	1^-	1^+	$(1, \infty)$	∞
$f''(x)$	0	-	$-\infty$	$+\infty$	+	0
$f'(x)$	0	-	$-\infty$	$-\infty$	-	0
$f(x)$	1	$f(-1) = 0$	$-\infty$	∞		1

We use this to put together a sketch of the graph: The horizontal line $y = 1$ and the vertical line $x = 1$ are called **asymptotes** of the graph.



Example 2: The graph of $f(x) = \frac{x}{x^2 - 1}$ is:

Step 1:

$$f'(x) = \frac{(x^2 - 1)(x)' - (x)(x^2 - 1)'}{(x^2 - 1)^2} = \frac{(x^2 - 1)(1) - (x)(2x)}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2}$$

$$\text{and } f''(x) = -\frac{(x^2 - 1)^2(x^2 + 1)' - (x^2 + 1)((x^2 - 1)^2)'}{((x^2 - 1)^2)^2} =$$

$$-\frac{(x^2 - 1)^2(2x) - (x^2 + 1)(2(x^2 - 1)(2x))}{(x^2 - 1)^4} = -2x \frac{(x^2 - 1) - (x^2 + 1)(2)}{(x^2 - 1)^3} =$$

$$-2x \frac{x^2 - 1 - 2x^2 - 2}{(x^2 - 1)^3} = -2x \frac{-3 - x^2}{(x^2 - 1)^3} = 2x \frac{3 + x^2}{(x^2 - 1)^3}$$

The only “interesting values” of f are -1 , 0 , and 1 .

Step 2: Put these values of x into increasing order. $-1, 0, 1$.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f .

x	$-\infty$	$(-\infty, -1)$	-1^-	-1^+	$(-1, 0)$	0	$(0, 1)$	1^-	1^+	$(1, \infty)$	∞
$f''(x)$	0	$-$	$-\infty$	$+\infty$	$+$	0	$-$	$-\infty$	∞	$+$	0
$f'(x)$	0	$-$	$-\infty$	$-\infty$	$-$	-1	$-$	$-\infty$	$-\infty$	$-$	0
$f(x)$	0	$-$	$-\infty$	∞	$+$	0	$-$	$-\infty$	∞	$+$	0

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.

We use this to put together a sketch of the graph:

The horizontal line $y = 0$ and the vertical lines $x = -1$ and $x = 1$ are called **asymptotes** of the graph.

