Sketching Polynomials

Recall that the degree of a polynomial is the highest power of the independent variable appearing in it. A polynomial can have no more roots than its degree, but it may have less. If the degree of a polynomial is odd it must have at leat one root.

The degree of the first derivative of a polynomial is one less than that of the polynomial, and that of the second derivative is two less.

Our skills in finding roots of polynomials and factoring them now become very powerful curve sketching tools.

Our first examples deal with degree 2 polynomials or quadratic functions, which have exactly one critical point and no inflection points. Their graphs are parabolas.

Example 1: Sketch the graph of $f(x) = 4 - x^2$.

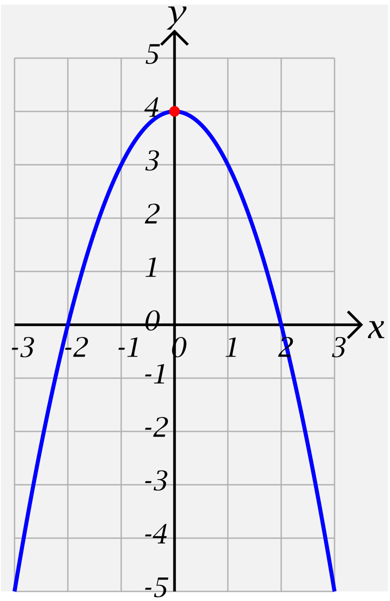
Step 1: $f'(x) = -2x$ and $f''(x) = -2 < 0$. The only "interesting value" of f is 0, which divides the domain $(-\infty, \infty)$ into two intervals: $(-\infty, 0)$ and $(0, \infty)$.

Step 2: Put these values of x into increasing order. 0

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	-2	-2	-2
$f'(x)$	$+$	0	$-$
$f(x)$		4	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 2:

Sketch the graph of $f(x) = (4 - x)^2$.

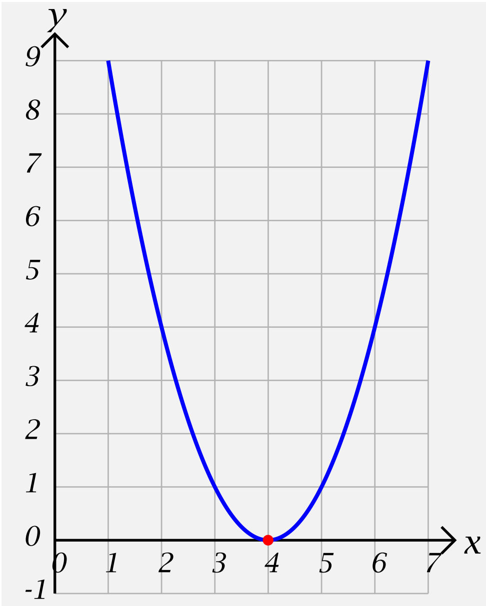
Step 1: $f'(x) = 2(4 - x)(-1) = 2(x - 4)$ and $f''(x) = 2 > 0$. The only "interesting value" of f is 4, which divides the domain $(-\infty, \infty)$ into two intervals: $(-\infty, 4)$ and $(4, \infty)$.

Step 2: Put these values of x into increasing order. 4

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 4)$	4	$(4, \infty)$
$f''(x)$	2	2	2
$f'(x)$	$-$	0	$+$
$f(x)$		0	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



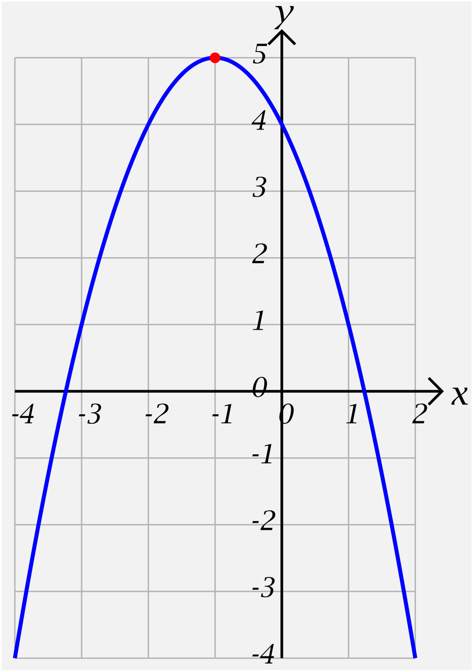
Example 3: Sketch the graph of $f(x) = 4 - 2x - x^2$.

Step 1: $f'(x) = -2 - 2x = -2(x - (-1))$ and $f''(x) = -2 < 0$. The only "interesting value" of f is -1, which divides the domain $(-\infty, \infty)$ into two intervals: $(-\infty, -1)$ and $(-1, \infty)$.

Step 2: Put these values of x into increasing order. -1

Step 3: Put together as good a table as you can.

x	$(-\infty, -1)$	-1	$(-1, \infty)$
$f''(x)$	-2	-2	-2
$f'(x)$	$+$	0	$-$
$f(x)$		5	



Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

Cubic Polynomials

Polynomials of degree 3 have exactly one inflection point, and either two, one, or no critical points. It is always possible to calculate these points using basic algebra.

Example 4: Sketch the graph of $f(x) = x^3 - 9x$.

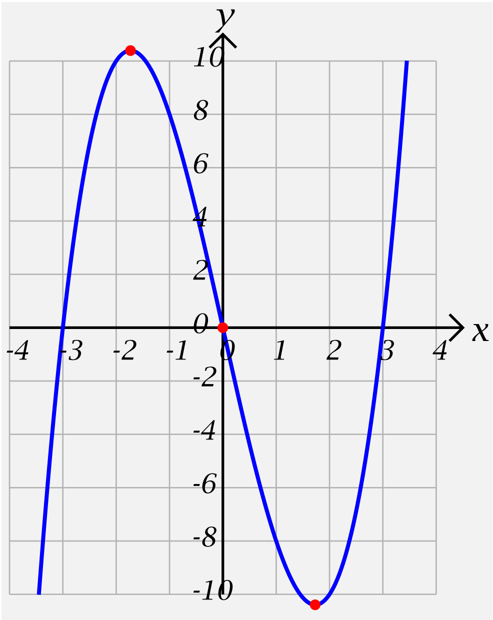
Step 1: $f'(x) = 3x^2 - 9$ and $f''(x) = 6x$. The "interesting values" of f' are $-\sqrt{3}$ and $\sqrt{3}$, and the interesting value of f'' is 0.

Step 2: Put these values of x into increasing order. $-\sqrt{3}, 0, \sqrt{3}$.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$		$6\sqrt{3}$		0		$-6\sqrt{3}$	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 5: Sketch the graph of $f(x) = x^3 + 9x$.

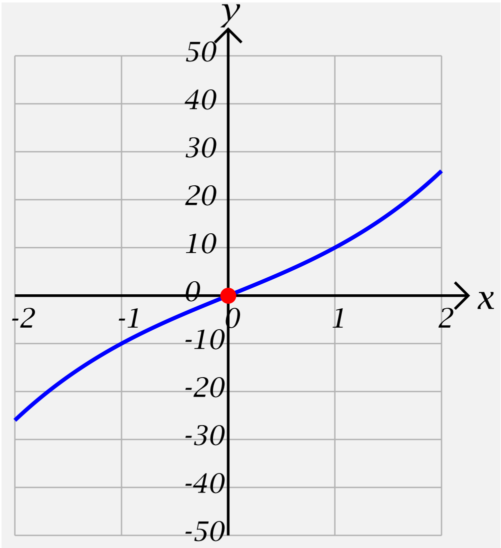
Step 1: $f'(x) = 3x^2 + 9 > 0$ and $f''(x) = 6x$. The only "interesting value" is 0.

Step 2: Put these values of x into increasing order. 0

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	0	$+$
$f'(x)$	$+$	$+$	$+$
$f(x)$		0	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 6: Sketch the graph of $f(x) = 2x^3 - 4x^2 + 2$.

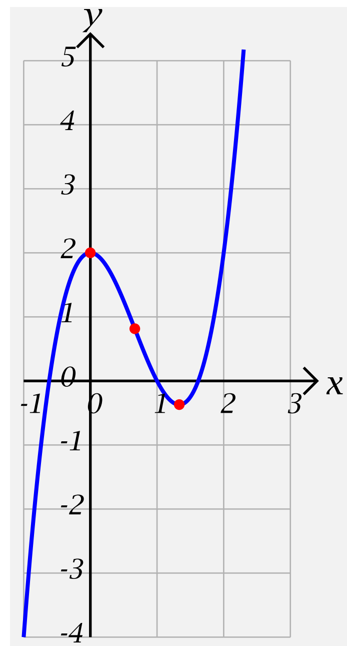
Step 1: $f'(x) = 6x^2 - 8x = 2x(3x-4)$ and $f''(x) = 12x - 8 = 4(3x-2)$. The "interesting values" of f' are 0 and $\frac{4}{3}$, and the interesting value of f'' is $\frac{2}{3}$.

Step 2: Put these values of x into increasing order. $0, \frac{2}{3}, \frac{4}{3}$.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0	$\left(0, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \frac{4}{3}\right)$	$\frac{4}{3}$	$\left(\frac{4}{3}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$		2		$\frac{22}{27}$		$-\frac{10}{27}$	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 7: Sketch the graph of $f(x) = 3x^3 - 3x^2 + 1$.

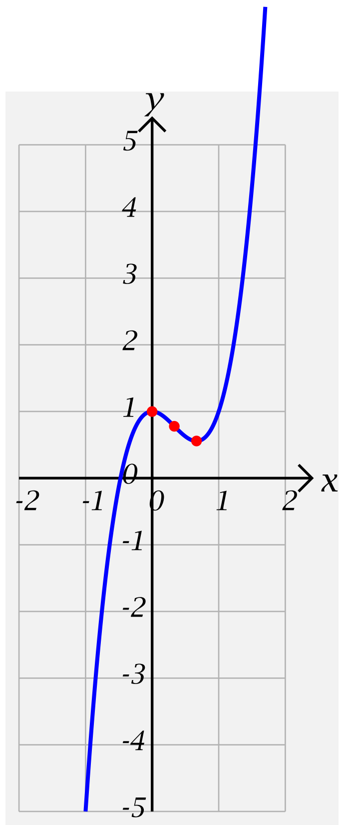
Step 1: $f'(x) = 9x^2 - 6x = 3x(3x-2)$ and $f''(x) = 18x - 6 = 6(3x-1)$. The "interesting values" of f' are 0 and $\frac{2}{3}$, and the interesting value of f'' is $\frac{1}{3}$.

Step 2: Put these values of x into increasing order. $0, \frac{1}{3}, \frac{2}{3}$.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 0)$	0	$\left(0, \frac{1}{3}\right)$	$\frac{1}{3}$	$\left(\frac{1}{3}, \frac{2}{3}\right)$	$\frac{2}{3}$	$\left(\frac{2}{3}, \infty\right)$
$f''(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$		1		$\frac{7}{9}$		$\frac{5}{9}$	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Quartic Polynomials

Polynomials of degree 4 have exactly at most two inflection points(possibly none), and up to three critical points. It is not always possible to calculate all of these points using basic algebra.

Example 8: Sketch the graph of $f(x) = \frac{1}{10}(x^4 - 18x^2 + 5)$.

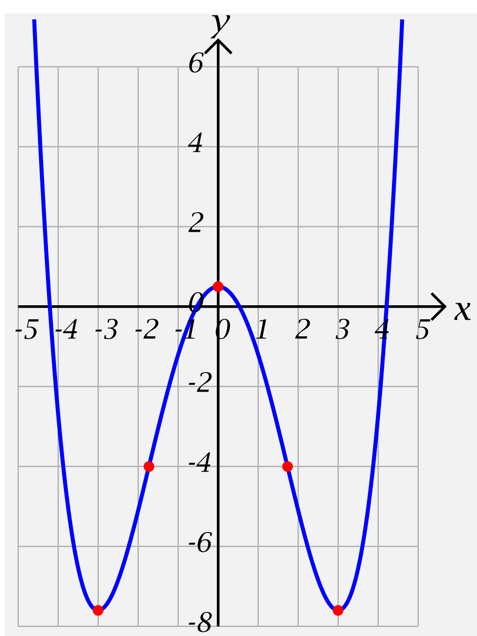
Step 1: $f'(x) = \frac{1}{10}(4x^3 - 36x) = \frac{4}{10}x(x^2 - 9) = \frac{4}{10}x(x-3)(x+3)$ and $f''(x) = \frac{1}{10}(12x^2 - 36) = \frac{12}{10}(x^2 - 3)$. The "interesting values" of f' are -3, 0 and 3, and the interesting values of f'' are $-\sqrt{3}$ and $\sqrt{3}$.

Step 2: Put these values of x into increasing order. $-3, -\sqrt{3}, 0, \sqrt{3}, 3.$

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, -3)$	-3	$(-3, -\sqrt{3})$	$-\sqrt{3}$	$(-\sqrt{3}, 0)$	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, 3)$	3	$(3, \infty)$
$f''(x)$				0				0			
$f'(x)$		0				0				0	
$f(x)$		-7.6		-2.2		.5		-2.2		-7.6	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.



Example 9: Sketch the graph of $y = f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution: We have $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1) = 0$ if $x = -1$, 0, or 2, and $f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2) = 0$ if

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6}$$

$= \dfrac{1 \pm \sqrt{7}}{3} \doteq -0.54, 1.22$, so the "interesting" values of f are $\{-1, 0, 2\}$ and $\left\{\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right\}$

In increasing order, these numbers are

$-1, \dfrac{1-\sqrt{7}}{3}, 0, \dfrac{1+\sqrt{7}}{3}$, and 2. We construct a table:

x	$(-\infty, -1)$	-1	$(-1, \frac{1-\sqrt{7}}{3})$	$\frac{1-\sqrt{7}}{3}$	$(\frac{1-\sqrt{7}}{3}, 0)$	0	$(0, \frac{1+\sqrt{7}}{3})$	$\frac{1+\sqrt{7}}{3}$	$(\frac{1+\sqrt{7}}{3}, 2)$	2	$(2, \infty)$
$f''(x)$	+	+	+	0	−	−	−	0	+	+	+
$f'(x)$	−	0	+	+	+	0	−	−	−	0	+
$f(x)$		0				5				−27