

# Sketching Products of Log and Exponential Functions

**Example 1:**  $f(x) = xe^x$

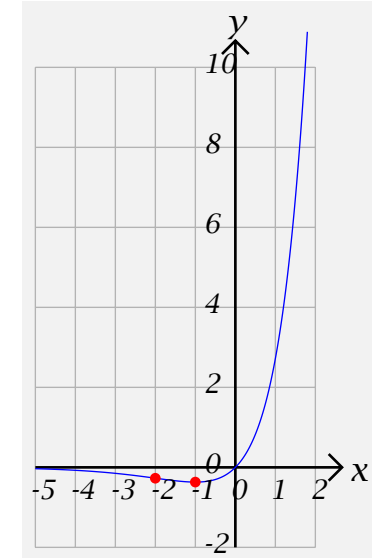
**Step 1:**  $f'(x) = (x)'e^x + x(e^x)' = (1)e^x + xe^x = (x+1)e^x$  and  $f''(x) = (x+1)'e^x + (x+1)(e^x)' = (1)e^x + (x+1)e^x = (x+2)e^x$ .  
The “interesting values” of  $f$  are  $-1$  and  $-2$ , which divide the domain  $(-\infty, \infty)$  into three intervals:  $(-\infty, -2)$ ,  $(-2, -1)$  and  $(-1, \infty)$ .

**Step 2:** Put these values of  $x$  into increasing order.  $-2, -1$

**Step 3:** Put together as good a table as you can.

$x$	$(-\infty, -2)$	$-2$	$(-2, -1)$	$-1$	$(-1, \infty)$
$f''(x)$	$-$	$0$	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$-$	$0$	$+$
$f(x)$		$-\frac{2}{e^2} \doteq -0.27$		$-\frac{1}{e} \doteq -0.36$	$f(0) = 0$

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 2:**  $f(x) = xe^{-2x}$

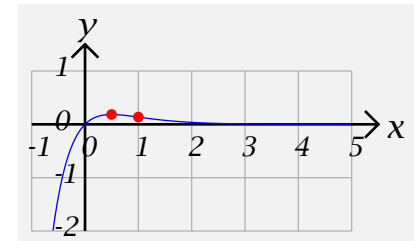
**Step 1:**  $f'(x) = (x)'e^{-2x} + x(e^{-2x})' = (1)e^{-2x} + xe^{-2x}(-2) = (-2x + 1)e^{-2x}$  and  $f''(x) = (-2x + 1)'e^{-2x} + (-2x + 1)(e^{-2x})' = (-2)e^{-2x} + (-2x + 1)e^{-2x}(-2) = (4x - 4)e^{-2x} = 4(x - 1)e^{-2x}$ . The “interesting values” of  $f$  are  $\frac{1}{2}$  and 1, which divide the domain  $(-\infty, \infty)$  into three intervals:  $(-\infty, \frac{1}{2})$ ,  $(\frac{1}{2}, 1)$  and  $(1, \infty)$ .

**Step 2:** Put these values of  $x$  into increasing order.  $\frac{1}{2}, 1$

**Step 3:** Put together as good a table as you can.

$x$	$(-\infty, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, 1)$	1	$(1, \infty)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	$f(0) = 0$	$\frac{1}{2e} \doteq 0.183$		$\frac{1}{e^2} \doteq 0.135$	

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 3:**  $f(x) = x^2e^{-2x}$

**Step 1:**  $f'(x) = (x^2)'e^{-2x} + x^2(e^{-2x})' = (2x)e^{-2x} + x^2e^{-2x}(-2) = (-2x^2 + 2x)e^{-2x} = -2x(x-1)$  and  $f''(x) = (-2x^2 + 2x)'e^{-2x} + (-2x^2 + 2x)(e^{-2x})' = (-4x + 2)e^{-2x} + (-2x^2 + 2x)e^{-2x}(-2) = (-4x + 2)e^{-2x} + (4x^2 - 4x)e^{-2x} = (4x^2 - 8x + 2)e^{-2x} = 2(2x^2 - 4x + 1)e^{-2x}$ .

The “interesting values” from  $f'$  are 0 and 1, and those of  $f''$  are:

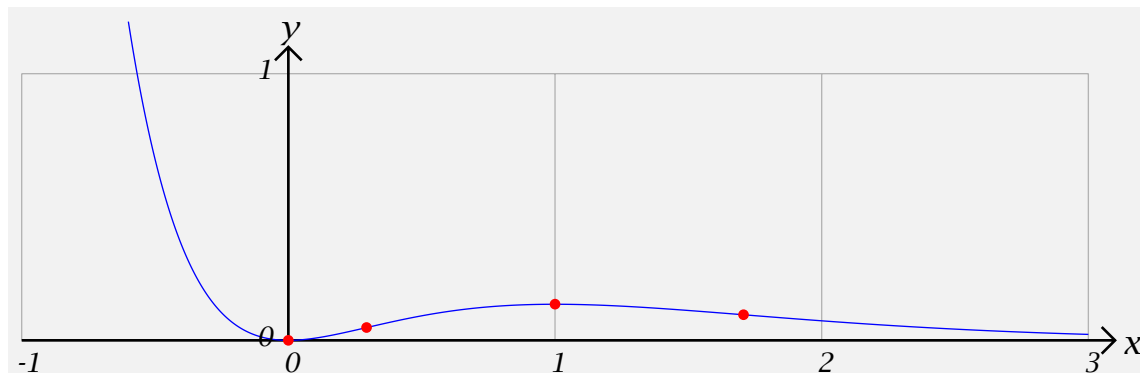
$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)1}}{2(2)} = \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2 \pm \sqrt{2}}{2} = 1 \pm \frac{\sqrt{2}}{2}.$$

**Step 2:** Put these values of  $x$  into increasing order.  $0, 1 - \frac{\sqrt{2}}{2}, 1, 1 + \frac{\sqrt{2}}{2}$

**Step 3:** Put together as good a table as you can.

$x$	$(-\infty, 0)$	0	$(0, 1 - \frac{\sqrt{2}}{2})$	$1 - \frac{\sqrt{2}}{2}$	$(1 - \frac{\sqrt{2}}{2}, 1)$	1	$(1, 1 + \frac{\sqrt{2}}{2})$	$1 + \frac{\sqrt{2}}{2}$	$(1 + \frac{\sqrt{2}}{2}, \infty)$
$f''(x)$	+	+	+	0	-	-	-	0	+
$f'(x)$	-	0	+	+	+	0	-	-	-
$f(x)$	+	0	+	+	+	$\frac{1}{e^2} \doteq 0.135$	+	+	+

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 4:**  $f(x) = x^2 e^{2x}$

**Step 1:**  $f'(x) = (x^2)'e^{2x} + x^2(e^{2x})' = (2x)e^{2x} + x^2(e^{2x})'(2) = (2x^2 + 2x)e^{2x} = 2x(x+1)e^{2x}$  and  $f''(x) = (2x^2 + 2x)'e^{2x} + (2x^2 + 2x)(e^{2x})'(2) = (4x + 2)e^{2x} + (2x^2 + 2x)e^{2x}(2) = (4x + 2)e^{2x} + (4x^2 + 4x)e^{2x} = (4x^2 + 8x + 2)e^{2x} = 2(2x^2 + 4x + 1)e^{2x}$ .

The “interesting values” from  $f'$  are 0 and 1, and those of  $f''$  are:

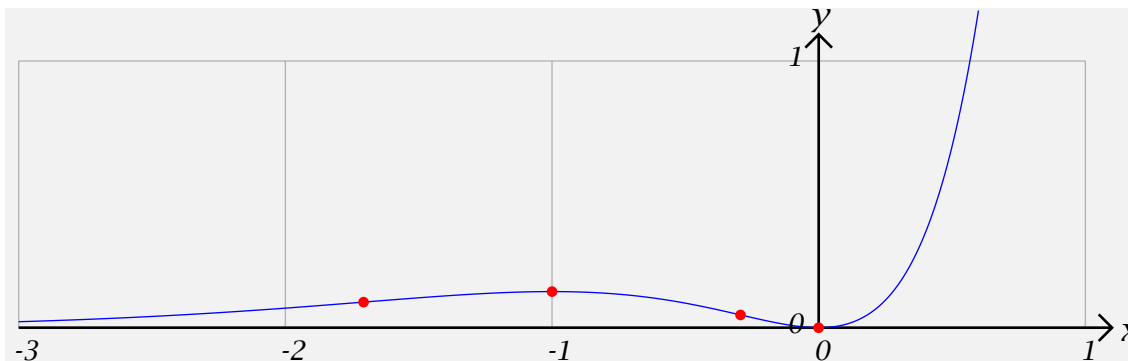
$$\frac{-(4) \pm \sqrt{(4)^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{16 - 8}}{4} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2} = -1 \pm \frac{\sqrt{2}}{2}.$$

**Step 2:** Put these values of  $x$  into increasing order.  $-1 - \frac{\sqrt{2}}{2}, -1, -1 + \frac{\sqrt{2}}{2}, 0$

**Step 3:** Put together as good a table as you can.

$x$	$\left(-\infty, \frac{-2 - \sqrt{2}}{2}\right)$	$-1 - \frac{\sqrt{2}}{2}$	$\left(-1 - \frac{\sqrt{2}}{2}, -1\right)$	$-1$	$\left(-1, -1 + \frac{\sqrt{2}}{2}\right)$	$-1 + \frac{\sqrt{2}}{2}$	$\left(-1 + \frac{\sqrt{2}}{2}, 0\right)$	$0$	$(0, \infty)$
$f''(x)$	+	0	-	-	-	0	+	+	+
$f'(x)$	+	+	+	0	-	-	-	0	+
$f(x)$	+	0	+	$\frac{1}{e^2}$	+	+	+	0	+

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 5:**  $f(x) = xe^{x^2}$

**Step 1:**  $f'(x) = (x)'e^{x^2} + x(e^{x^2})' = (1)e^{x^2} + xe^{x^2}(2x) = (2x^2 + 1)e^{x^2} > 0$  and  $f''(x) = (2x^2 + 1)'e^{x^2} + (2x^2 + 1)(e^{x^2})' = (4x)e^{x^2} + (2x^2 + 1)e^{x^2}(2x) = (4x)e^{x^2} + (4x^3 + 2x)e^{x^2} = (4x^3 + 6x)e^{x^2} = 2x(2x^2 + 3)e^{x^2}$ .

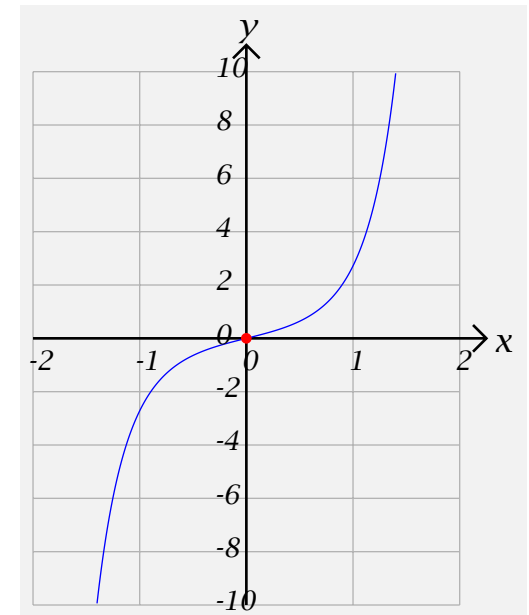
The only “interesting values” is 0.

**Step 2:** Put these values of  $x$  into increasing order. **0**

**Step 3:** Put together as good a table as you can.

$x$	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	-	0	-+
$f'(x)$	+	+	+ -
$f(x)$	-	0	+

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 6:**  $f(x) = e^{-\frac{1}{x^2}}$

**Step 1:**  $f'(x) = e^{-\frac{1}{x^2}} \left(-\frac{1}{x^2}\right)' = e^{-\frac{1}{x^2}} (-(-2)x^{-3}) = 2x^{-3}e^{-\frac{1}{x^2}}$  and

$$f''(x) = (2x^{-3})' e^{-\frac{1}{x^2}} + (2x^{-3}) \left(e^{-\frac{1}{x^2}}\right)' = (2(-3)x^{-4}) e^{-\frac{1}{x^2}} + (2x^{-3}) e^{-\frac{1}{x^2}} \left(-\frac{1}{x^2}\right)' =$$

$$(-6x^{-4}) e^{-\frac{1}{x^2}} + (2x^{-3}) e^{-\frac{1}{x^2}} (2x^{-3}) = \left(\frac{-6}{x^4} + \frac{4}{x^6}\right) e^{-\frac{1}{x^2}} = \left(\frac{-6x^2 + 4}{x^6}\right) e^{-\frac{1}{x^2}} = -6x^{-6} \left(x^2 - \frac{2}{3}\right) e^{-\frac{1}{x^2}}.$$

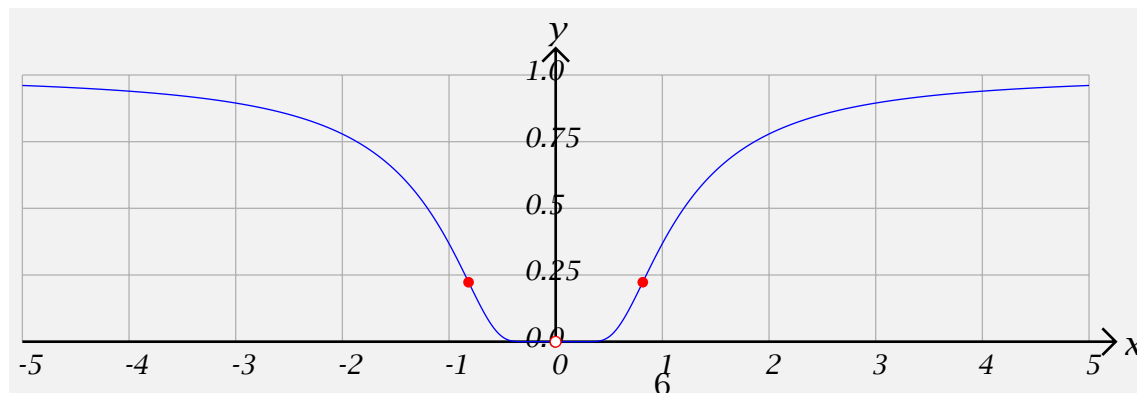
The “interesting values” 0 and  $\pm\sqrt{\frac{2}{3}}$ .

**Step 2:** Put these values of  $x$  into increasing order.  $-\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}}$

**Step 3:** Put together as good a table as you can.

$x$	$\left(-\infty, -\sqrt{\frac{2}{3}}\right)$	$-\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, 0\right)$	0	$\left(0, \sqrt{\frac{2}{3}}\right)$	$\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, \infty\right)$
$f''(x)$	+	0	-	und	-	0	+
$f'(x)$	-	-	-	und	+	+	+
$f(x)$	$\leq 1$	$e^{-\frac{3}{2}} \doteq 0.22$	+	und	+	$e^{-\frac{3}{2}} \doteq 0.22$	$\leq 1$

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 7:**  $f(x) = x \ln x$

**Step 1:** The domain of  $f$  is  $(0, \infty)$ .  $f'(x) = (x)' \ln x + x(\ln x)' = (1) \ln x + x \frac{1}{x} = \ln x + 1 = 0$  if  $\ln x = -1$ . Taking exponentials, we get  $x = e^{\ln x} = e^{-1} = \frac{1}{e}$

Also  $f''(x) = \frac{1}{x}$ .

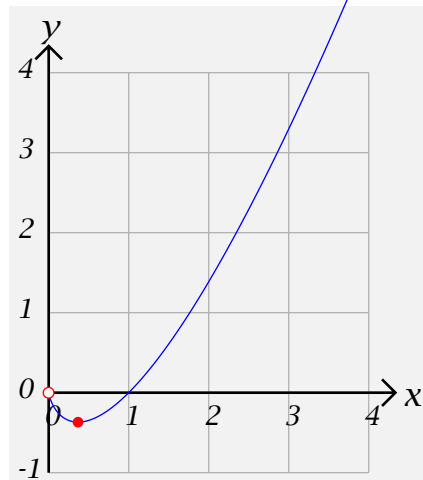
The “interesting values” are 0 and  $\frac{1}{e}$ .

**Step 2:** Put these values of  $x$  into increasing order.  $0, \frac{1}{e}$

**Step 3:** Put together as good a table as you can.

$x$	$(0, \frac{1}{e})$	$\frac{1}{e}$	$(\frac{1}{e}, \infty)$
$f''(x)$	+	+	+
$f'(x)$	-	0	+
$f(x)$	-	$-\frac{1}{e} \doteq -0.37$	$f(1) = 0$

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 8:**  $f(x) = x^2 \ln x$

**Step 1:** The domain of  $f$  is  $(0, \infty)$ .  $f'(x) = (x^2)' \ln x + (x^2)(\ln x)' = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1) = 0$  if  $x = 0$  or  $\ln x = -\frac{1}{2}$ . Taking exponentials, we get  $x = e^{\ln x} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

Also  $f''(x) = (2x)' \ln x + (2x)(\ln x)' + 1 = 2 \ln x + 2x \frac{1}{x} + 1 = 2 \ln x + 3 = 0$  if  $\ln x = -\frac{3}{2}$ .

Taking exponentials, we get  $x = e^{\ln x} = e^{-\frac{3}{2}} = \frac{1}{e\sqrt{e}}$ .

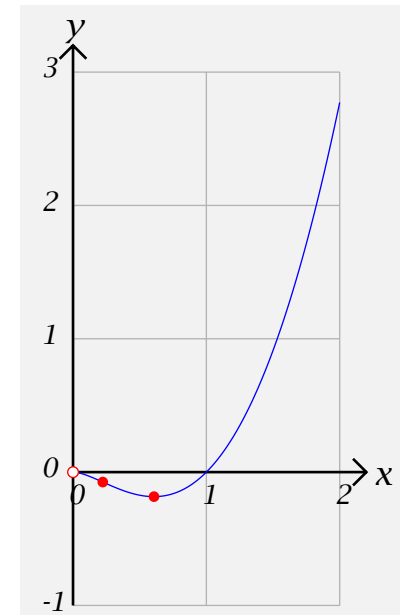
The “interesting values” are  $0$ ,  $\frac{1}{\sqrt{e}}$ , and  $\frac{1}{e\sqrt{e}}$ .

**Step 2:** Put these values of  $x$  into increasing order.  $0, \frac{1}{e\sqrt{e}}, \frac{1}{\sqrt{e}}$

**Step 3:** Put together as good a table as you can.

$x$	$(0, \frac{1}{e\sqrt{e}})$	$\frac{1}{e\sqrt{e}}$	$(\frac{1}{e\sqrt{e}}, \frac{1}{\sqrt{e}})$	$\frac{1}{\sqrt{e}}$	$(\frac{1}{\sqrt{e}}, \infty)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	-	$-\frac{3}{2e^3} \doteq -0.07$		$-\frac{1}{2e} \doteq -0.18$	$f(1) = 0$

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 9:**  $f(x) = \frac{\ln x}{x^2}$

**Step 1:** The domain of  $f$  is  $(0, \infty)$ .

$$f'(x) = \frac{(x^2)(\ln x)' - (\ln x)(x^2)'}{(x^2)^2} = \frac{(x^2)\left(\frac{1}{x}\right) - (\ln x)(2x)}{x^4} = \frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} = 0 \text{ if } \ln x = \frac{1}{2}. \text{ Taking exponentials, we get } x = e^{\ln x} = e^{\frac{1}{2}} = \sqrt{e}$$

$$\text{Also } f''(x) = \frac{(x^3)(1 - 2 \ln x)' - (1 - 2 \ln x)(x^3)'}{(x^3)^2} = \frac{(x^3)\left(-2\frac{1}{x}\right) - (1 - 2 \ln x)(3x^2)}{x^6} = \frac{-2x^2 - (1 - 2 \ln x)(3x^2)}{x^6} = \frac{-2 - (1 - 2 \ln x)(3)}{x^4} = \frac{6 \ln x - 5}{x^4} = 0 \text{ if } \ln x = \frac{5}{6}.$$

Taking exponentials, we get  $x = e^{\ln x} = e^{\frac{5}{6}}$ .

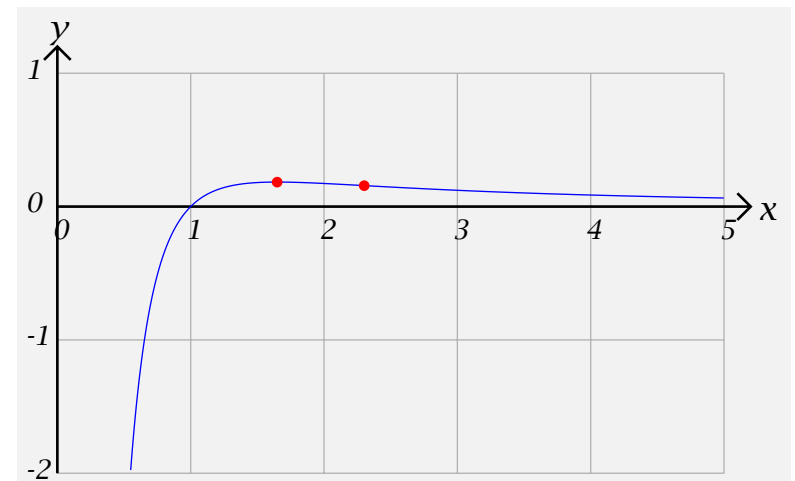
The “interesting values” are  $0$ ,  $e^{\frac{1}{2}}$ , and  $e^{\frac{5}{6}}$ .

**Step 2:** Put these values of  $x$  into increasing order.  $0, e^{\frac{1}{2}}, e^{\frac{5}{6}}$

**Step 3:** Put together as good a table as you can.

$x$	$(0, e^{\frac{1}{2}})$	$e^{\frac{1}{2}}$	$(e^{\frac{1}{2}}, e^{\frac{5}{6}})$	$e^{\frac{5}{6}}$	$(e^{\frac{5}{6}}, \infty)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	-	$\frac{1}{2e} \doteq 1.36$		$\frac{5}{6e^{\frac{5}{3}}} \doteq 0.16$	$f(1) = 0$

**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



**Example 10:**  $f(x) = \frac{\ln x}{x^{\frac{1}{2}}}$

**Step 1:** The domain of  $f$  is  $(0, \infty)$ .

$$f'(x) = \frac{(x^{\frac{1}{2}})(\ln x)' - (\ln x)(x^{\frac{1}{2}})'}{(x^{\frac{1}{2}})^2} = \frac{(x^{\frac{1}{2}})\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x} = \frac{\frac{1}{x^{\frac{3}{2}}} - \frac{\frac{1}{2}\ln x}{x^{\frac{3}{2}}}}{x} =$$

$$\frac{2 - \ln x}{2x^{\frac{3}{2}}} = 0 \text{ if } \ln x = 2 \text{ or } x = e^2$$

$$\text{Also } f''(x) = \frac{(2x^{\frac{3}{2}})(2 - \ln x)' - (2 - \ln x)(2x^{\frac{3}{2}})'}{(2x^{\frac{3}{2}})^2} = \frac{(2x^{\frac{3}{2}})\left(-\frac{1}{x}\right) - (2 - \ln x)\left(2\frac{3}{2}x^{\frac{1}{2}}\right)}{4x^3} =$$

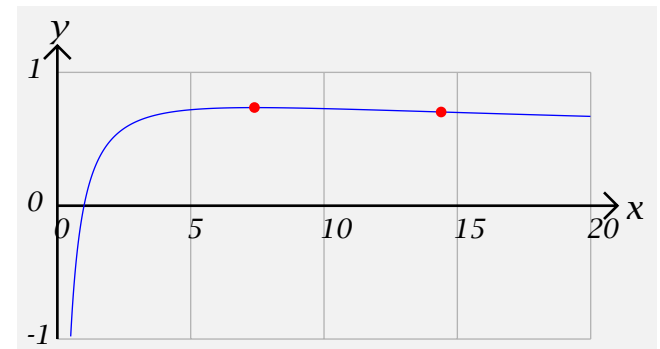
$$\frac{-2x^{\frac{1}{2}} - (2 - \ln x)(3x^{\frac{1}{2}})}{4x^3} = \frac{-2 - 3(2 - \ln x)}{4x^{\frac{5}{2}}} = \frac{3\ln x - 8}{4x^{\frac{5}{2}}} = 0 \text{ if } \ln x = \frac{8}{3} \text{ or } x = e^{\frac{8}{3}}$$

The “interesting values” are  $0$ ,  $e^2$ , and  $e^{\frac{8}{3}}$ .

**Step 2:** Put these values of  $x$  into increasing order.  $0, e^2, e^{\frac{8}{3}}$

**Step 3:** Put together as good a table as you can.

$x$	$(0, e^2)$	$e^2$	$(e^2, e^{\frac{8}{3}})$	$e^{\frac{8}{3}}$	$(e^{\frac{8}{3}}, \infty)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	$f(1) = 0$	$\frac{2}{e} \doteq 0.74$		$\frac{8}{3e^{\frac{4}{3}}} \doteq 0.70$	



**Step 4:** Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.