Sketching Graphs of Functions

Increasing and Decreasing Functions

Definition: A function f is increasing on an interval (a, b) if $f(x) \leq f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.

If f' exists on the interval, we have $f'(x) \geq 0$ for all x in (a, b).

Definition: A function f is strictly increasing on an interval (a, b) if $f(x) < f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.
If f' exists on the interval, we have $f'(x) > 0$ for all x in (a, b).

Definition: A function f is decreasing on an interval (a, b) if $f(x) \geq f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.

If f' is exists on the interval, we have $f'(x) \leq 0$ for all x in (a, b).

Definition: A function f is strictly decreasing on an interval (a, b) if $f(x) > f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.
If f' exists on the interval, we have $f'(x) < 0$ for all x in (a, b).

Second Derivatives

The derivative of the derivative of a function f is called its second derivative. We write

$f''(x) = (f'(x))'$ or $y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$

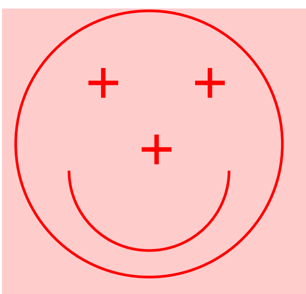
Example: If $f(x) = x^3 + x$ then

$f'(x) = 3x^2 + 1$, and $f''(x) = (3x^2 + 1)' = 6x$.

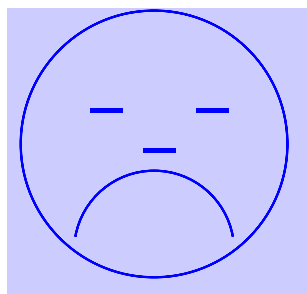
The actual numerical value of $f''(x)$ will not be important in this course, we will only be interested in its sign.

Definition: A function f is concave up on an interval (a, b) if its graph lies above the tangent line to the graph of $y = f(x)$ at every point of the interval.
We have $f''(x) \geq 0$ on such an interval.

Definition: A function f is concave down on an interval (a, b) if its graph lies below the tangent line to the graph of $y = f(x)$ at every point of the interval.
We have $f''(x) \leq 0$ on such an interval.



$f''(x) \geq 0$



$f''(x) \leq 0$

Definition: Any value in the domain of a function where the direction of concavity changes is called an inflection point. Such points are characterized by the second derivative changing sign at them.

It is good policy to label all extrema and inflection points when sketching the graph of a function.

An Organized Approach to Sketching the Graph of $y = f(x)$:

Step 1: Compute $f'(x)$ and $f''(x)$ and then find all of the "interesting values" of f: all values of x for which $f'(x)$ or $f''(x)$ equal 0 or are undefined.

Step 2: Put these values of x into increasing order.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.