

Sketching Products of Log and Exponential Functions

Example 1: $f(x) = xe^x$

Step 1: $f'(x) = (x)e^x + x(e^x) = (1)e^x + xe^x = (x+1)e^x$

and

$$f''(x) = (x+1)e^x + (x+1)(e^x) = (1)e^x + (x+1)e^x = (x+2)e^x.$$

The “interesting values” of f are -1 and -2 , which divide the domain $(-\infty, \infty)$ into three intervals: $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.

Step 2: Put these values of x into increasing order.

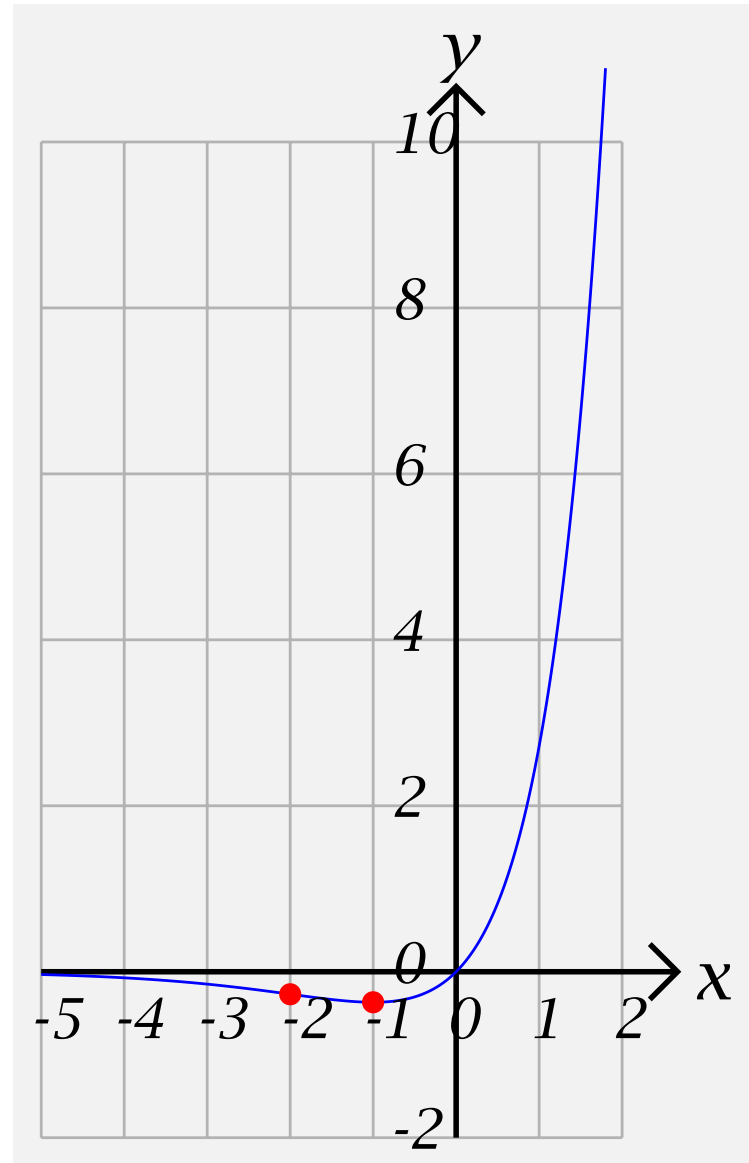
$-2, -1$

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Step 3: Put together as good a table as you can.

x	$(-\infty, -2)$	-2	$(-2, -1)$	-1	$(-1, \infty)$
$f''(x)$	$-$	0	$+$	$+$	$+$
$f'(x)$	$-$	$-$	$-$	0	$+$
$f(x)$		$-\frac{2}{e^2} \doteq -0.27$		$-\frac{1}{e} \doteq -0.36$	$f(0) = 0$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 2: $f(x) = xe^{-2x}$

Step 1: $f'(x) = (x)e^{-2x} + x(e^{-2x}) =$
 $(1)e^{-2x} + xe^{-2x}(-2) = (-2x + 1)e^{-2x}$ and
 $f''(x) = (-2x + 1)e^{-2x} + (-2x + 1)(e^{-2x}) =$
 $(-2)e^{-2x} + (-2x + 1)e^{-2x}(-2) = (4x - 4)e^{-2x} = 4(x - 1)e^{-2x}$.

The “interesting values” of f are $\frac{1}{2}$ and 1, which divide the domain $(-\infty, \infty)$ into three intervals: $(-\infty, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and $(1, \infty)$.

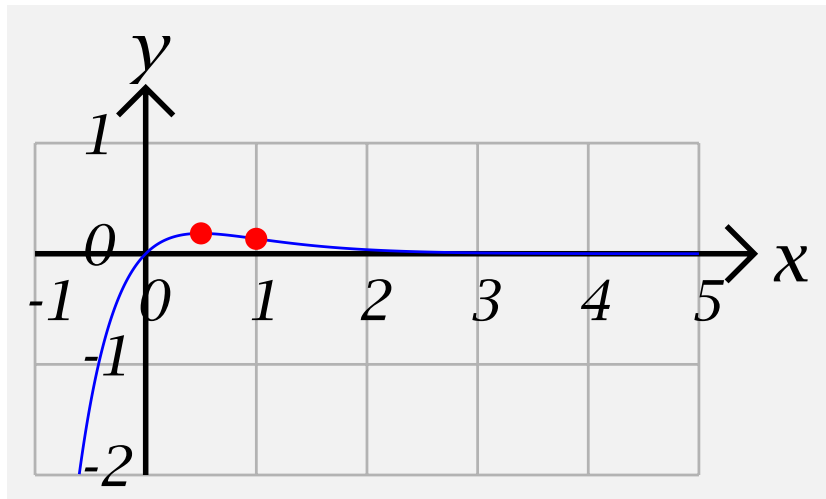
Step 2: Put these values of x into increasing order. $\frac{1}{2}, 1$

Step 3: Put together as good a table as you can.

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x	$\left(-\infty, \frac{1}{2}\right)$	$\frac{1}{2}$	$\left(\frac{1}{2}, 1\right)$	1	$(1, \infty)$
$f''(x)$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	$-$	$-$
$f(x)$	$f(0) = 0$	$\frac{1}{2e} \doteq 0.183$		$\frac{1}{e^2} \doteq 0.135$	

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 3: $f(x) = x^2 e^{-2x}$

Step 1: $f'(x) = (x^2)e^{-2x} + x^2(e^{-2x}) =$
 $(2x)e^{-2x} + x^2e^{-2x}(-2) = (-2x^2 + 2x)e^{-2x} = -2x(x - 1)$ and
 $f''(x) = (-2x^2 + 2x)e^{-2x} + (-2x^2 + 2x)(e^{-2x}) =$
 $(-4x + 2)e^{-2x} + (-2x^2 + 2x)e^{-2x}(-2) = (-4x + 2)e^{-2x} +$
 $(4x^2 - 4x)e^{-2x} = (4x^2 - 8x + 2)e^{-2x} = 2(2x^2 - 4x + 1)e^{-2x}.$

The “interesting values” from f' are 0 and 1, and those of f'' are:

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)1}}{2(2)} = \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} =$$

$$\frac{2 \pm \sqrt{2}}{2} = 1 \pm \frac{\sqrt{2}}{2}.$$

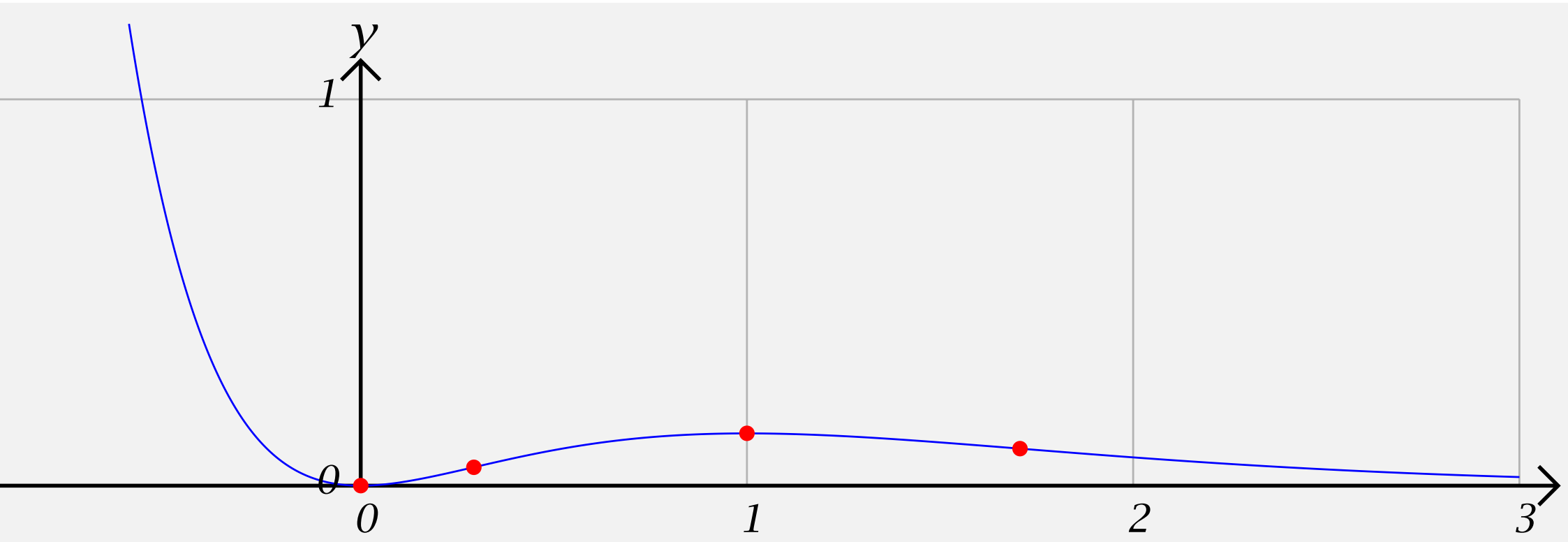
Step 2: Put these values of x into increasing order.

$$0, 1 - \frac{\sqrt{2}}{2}, 1, 1 + \frac{\sqrt{2}}{2}$$

Step 3: Put together as good a table as you can.

x	$(-\infty, 0)$	0	$\left(0, 1 - \frac{\sqrt{2}}{2}\right)$	$1 - \frac{\sqrt{2}}{2}$	$\left(1 - \frac{\sqrt{2}}{2}, 1\right)$	1	$(1, \infty)$
$f''(x)$	+	+	+	0	-	-	
$f'(x)$	-	0	+	+	+	0	
$f(x)$	+	0	+	+	+	$\frac{1}{e^2} \doteq 0.135$	

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 4: $f(x) = x^2 e^{2x}$

Step 1: $f'(x) = (x^2)e^{2x} + x^2(e^{2x}) =$
 $(2x)e^{2x} + x^2 e^{2x}(2) = (2x^2 + 2x)e^{2x} = 2x(x + 1)e^{2x}$ and
 $f''(x) = (2x^2 + 2x)e^{2x} + (2x^2 + 2x)(e^{2x}) =$
 $(4x + 2)e^{2x} + (2x^2 + 2x)e^{2x}(2) = (4x + 2)e^{-2x} + (4x^2 + 4x)e^{2x} =$
 $(4x^2 + 8x + 2)e^{2x} = 2(2x^2 + 4x + 1)e^{2x}.$

The “interesting values” from f' are 0 and 1, and those of f'' are:

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{16 - 8}}{4} = \frac{-4 \pm \sqrt{8}}{4} =$$

$$\frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2} = -1 \pm \frac{\sqrt{2}}{2}.$$

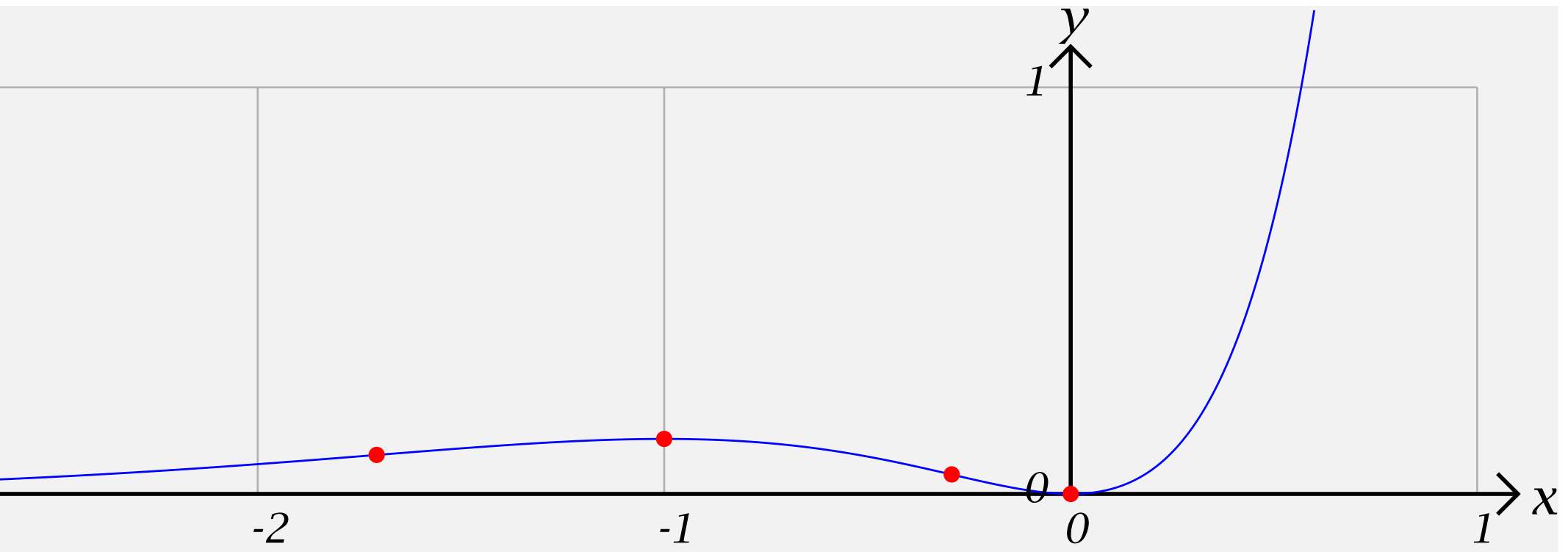
Step 2: Put these values of x into increasing order.

$$-1 - \frac{\sqrt{2}}{2}, -1, -1 + \frac{\sqrt{2}}{2}, 0$$

Step 3: Put together as good a table as you can.

x	$\left(-\infty, \frac{-2 - \sqrt{2}}{2}\right)$	$-1 - \frac{\sqrt{2}}{2}$	$\left(-1 - \frac{\sqrt{2}}{2}, -1\right)$	-1	$\left(-1, -1 + \frac{\sqrt{2}}{2}\right)$
$f''(x)$	+	0	-	-	-
$f'(x)$	+	+	+	0	-
$f(x)$	+	0	+	$\frac{1}{e^2}$	+

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 5: $f(x) = xe^{x^2}$

Step 1:

$$f'(x) = (x)e^{x^2} + x(e^{x^2}) = (1)e^{x^2} + xe^{x^2}(2x) = (2x^2 + 1)e^{x^2} > 0$$

$$\begin{aligned} \text{and } f''(x) &= (2x^2 + 1)e^{x^2} + (2x^2 + 1)(e^{x^2}) = \\ &= (4x)e^{x^2} + (2x^2 + 1)e^{x^2}(2x) = (4x)e^{x^2} + (4x^3 + 2x)e^{x^2} = \\ &= (4x^3 + 6x)e^{x^2} = 2x(2x^2 + 3)e^{x^2}. \end{aligned}$$

The only “interesting values” is 0.

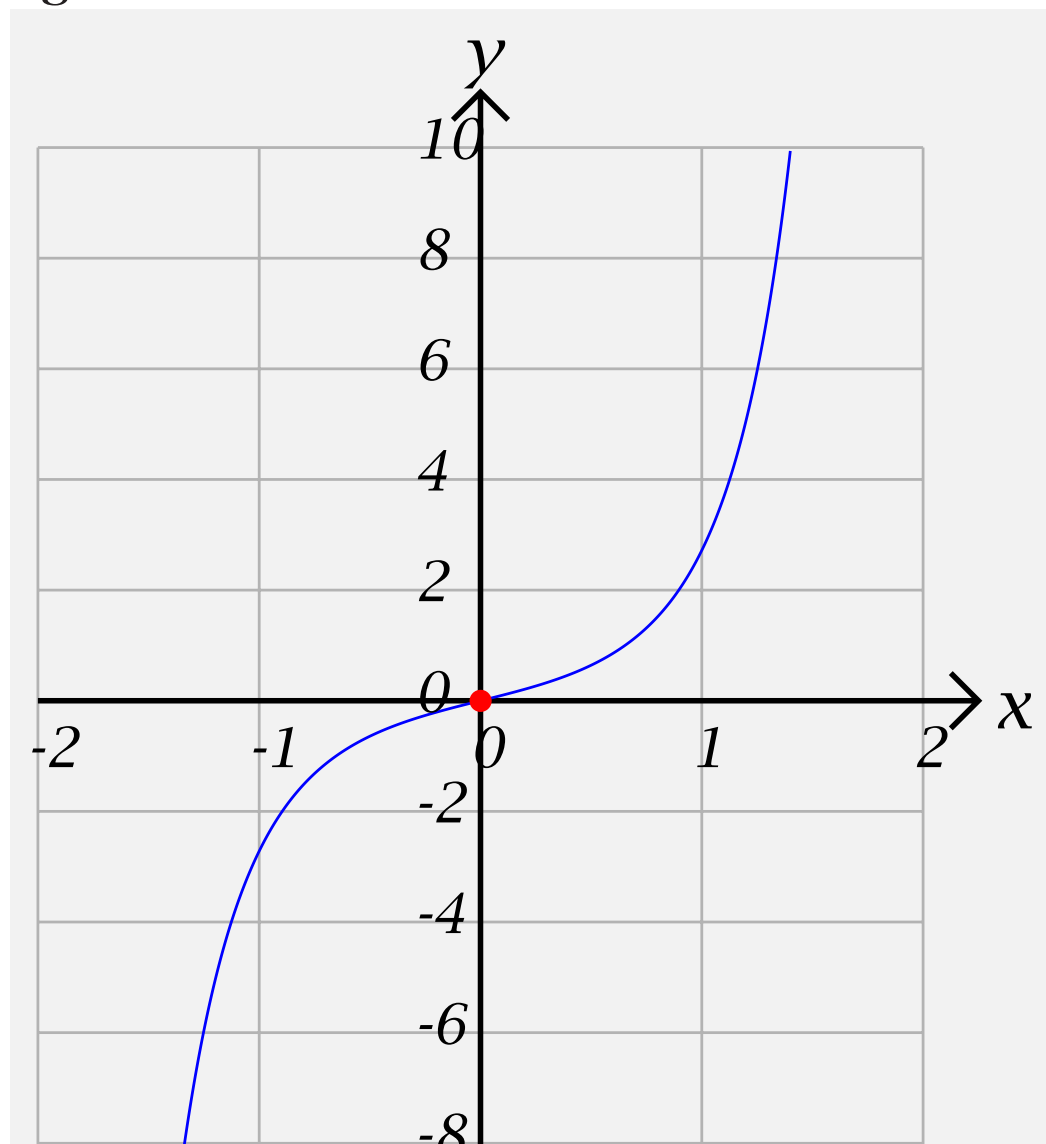
Step 2: Put these values of x into increasing order. **0**

Step 3: Put together as good a table as you can.

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x	$(-\infty, 0)$	0	$(0, \infty)$
$f''(x)$	$-$	0	$-+$
$f'(x)$	$+$	$+$	$+ -$
$f(x)$	$-$	0	$+$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 6: $f(x) = e^{-\frac{1}{x^2}}$

Step 1:

$$f'(x) = e^{-\frac{1}{x^2}} \left(-\frac{1}{x^2} \right)' = e^{-\frac{1}{x^2}} \left(-(-2)x^{-3} \right) = 2x^{-3}e^{-\frac{1}{x^2}} \text{ and}$$

$$f''(x) = (2x^{-3})e^{-\frac{1}{x^2}} + (2x^{-3}) \left(e^{-\frac{1}{x^2}} \right)' =$$

$$(2(-3)x^{-4})e^{-\frac{1}{x^2}} + (2x^{-3})e^{-\frac{1}{x^2}} \left(-\frac{1}{x^2} \right)' =.$$

$$(-6x^{-4})e^{-\frac{1}{x^2}} + (2x^{-3})e^{-\frac{1}{x^2}} (2x^{-3}) = \left(\frac{-6}{x^4} + \frac{4}{x^6} \right) e^{-\frac{1}{x^2}} =$$

$$\left(\frac{-6x^2 + 4}{x^6} \right) e^{-\frac{1}{x^2}} = -6x^{-6} \left(x^2 - \frac{2}{3} \right) e^{-\frac{1}{x^2}}.$$

The “interesting values” 0 and $\pm\sqrt{\frac{2}{3}}$.

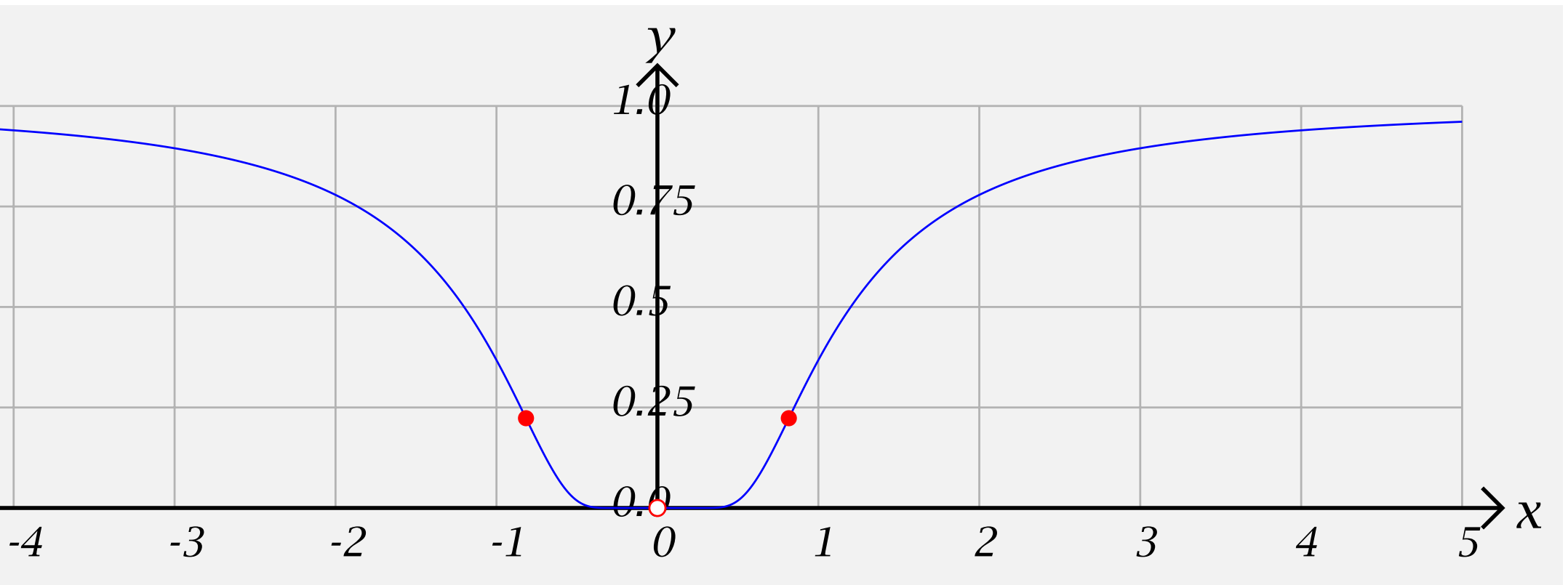
Step 2: Put these values of x into increasing order.

$$-\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}}$$

Step 3: Put together as good a table as you can.

x	$\left(-\infty, -\sqrt{\frac{2}{3}}\right)$	$-\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, 0\right)$	0	$\left(0, \sqrt{\frac{2}{3}}\right)$	$\sqrt{\frac{2}{3}}$
$f''(x)$	$+$	0	$-$	<i>und</i>	$-$	0
$f'(x)$	$-$	$-$	$-$	<i>und</i>	$+$	$+$
$f(x)$	≤ 1	$e^{-\frac{3}{2}} \doteq 0.22$	$+$	<i>und</i>	$+$	$e^{-\frac{3}{2}} \doteq 0.22$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 7: $f(x) = x \ln x$

Step 1: The domain of f is $(0, \infty)$.

$$f'(x) = (x) \ln x + x(\ln x) = (1) \ln x + x \frac{1}{x} = \ln x + 1 = 0 \text{ if}$$

$$\ln x = -1. \text{ Taking exponentials, we get } x = e^{\ln x} = e^{-1} = \frac{1}{e}$$

$$\text{Also } f''(x) = \frac{1}{x}.$$

The “interesting values” are 0 and $\frac{1}{e}$.

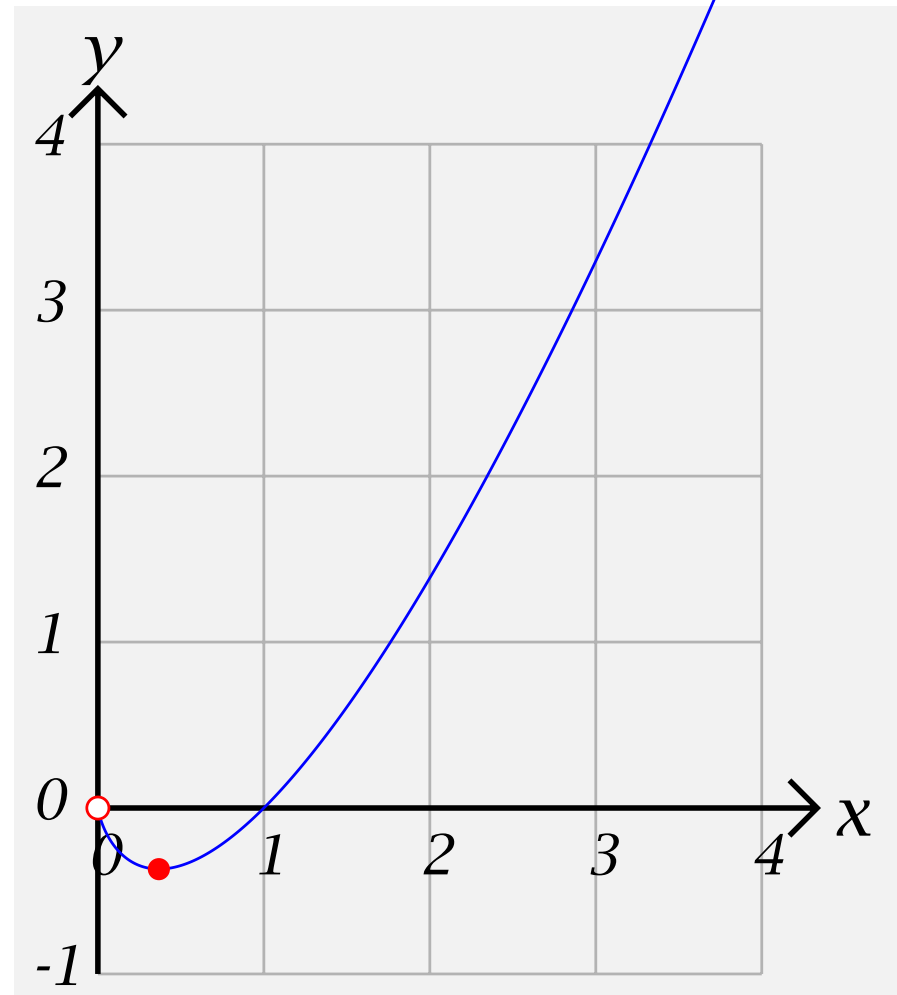
Step 2: Put these values of x into increasing order. $0, \frac{1}{e}$

Step 3: Put together as good a table as you can.

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x	$\left(0, \frac{1}{e}\right)$	$\frac{1}{e}$	$\left(\frac{1}{e}, \infty\right)$
$f''(x)$	+	+	+
$f'(x)$	-	0	+
$f(x)$	-	$-\frac{1}{e} \doteq -0.37$	$f(1) = 0$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 8: $f(x) = x^2 \ln x$

Step 1: The domain of f is $(0, \infty)$.

$$f'(x) = (x^2) \ln x + (x^2)(\ln x) = 2x \ln x + x^2 \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1) = 0 \text{ if } x = 0 \text{ or } \ln x = -\frac{1}{2}. \text{ Taking exponentials, we get } x = e^{\ln x} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$\text{Also } f''(x) = (2x) \ln x + (2x)(\ln x) + 1 = 2 \ln x + 2x \frac{1}{x} + 1 = 2 \ln x + 3 = 0 \text{ if } \ln x = -\frac{3}{2}.$$

$$\text{Taking exponentials, we get } x = e^{\ln x} = e^{-\frac{3}{2}} = \frac{1}{e\sqrt{e}}.$$

The “interesting values” are 0 , $\frac{1}{\sqrt{e}}$, and $\frac{1}{e\sqrt{e}}$.

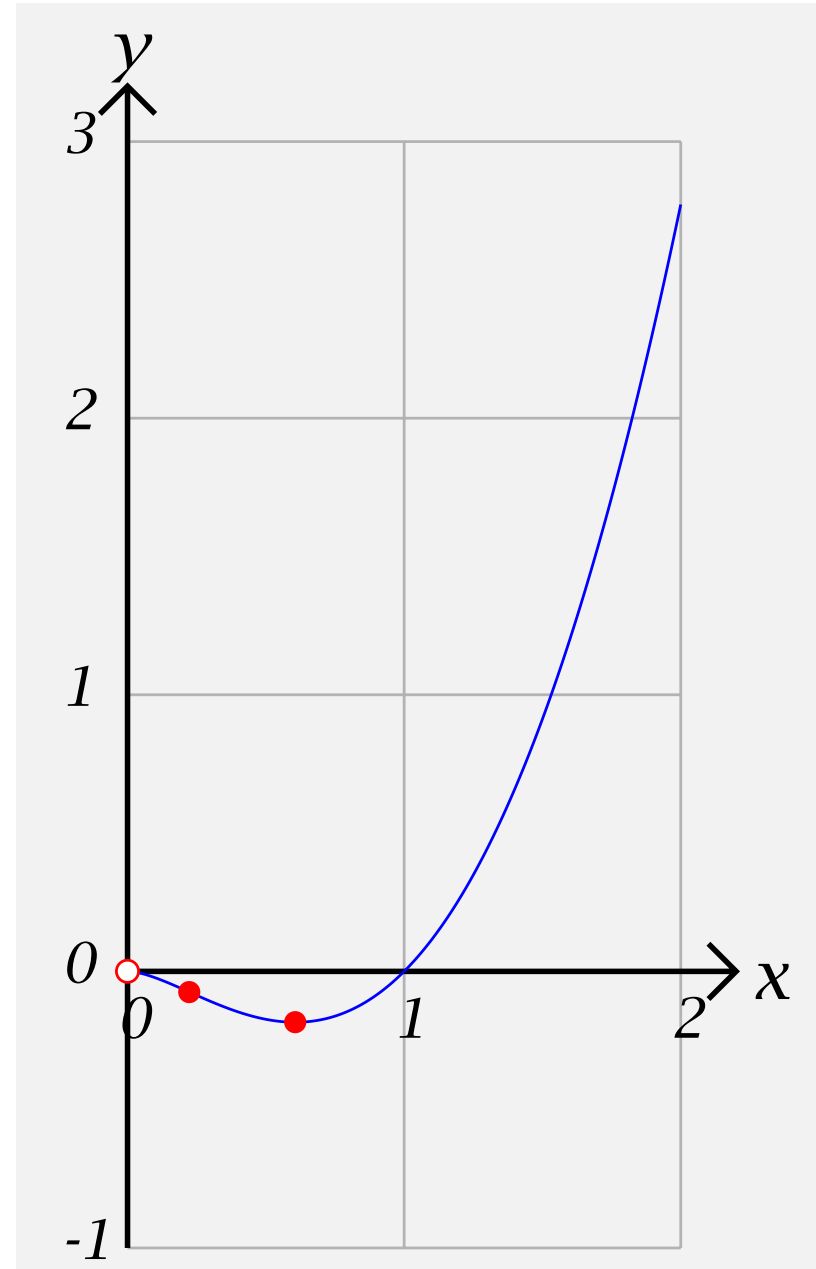
Step 2: Put these values of x into increasing order.

$$0, \frac{1}{e\sqrt{e}}, \frac{1}{\sqrt{e}}$$

Step 3: Put together as good a table as you can.

x	$\left(0, \frac{1}{e\sqrt{e}}\right)$	$\frac{1}{e\sqrt{e}}$	$\left(\frac{1}{e\sqrt{e}}, \frac{1}{\sqrt{e}}\right)$	$\frac{1}{\sqrt{e}}$	$\left(\frac{1}{\sqrt{e}}, \infty\right)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	-	$-\frac{3}{2e^3} \doteq -0.07$		$-\frac{1}{2e} \doteq -0.18$	$f(1) = 0$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 9: $f(x) = \frac{\ln x}{x^2}$

Step 1: The domain of f is $(0, \infty)$.

$$f'(x) = \frac{(x^2)(\ln x) - (\ln x)(x^2)}{(x^2)^2} = \frac{(x^2)\left(\frac{1}{x}\right) - (\ln x)(2x)}{x^4} =$$

$$\frac{x - 2x \ln x}{x^4} = \frac{1 - 2 \ln x}{x^3} = 0 \text{ if } \ln x = \frac{1}{2}. \text{ Taking exponentials,}$$

we get $x = e^{\ln x} = e^{\frac{1}{2}} = \sqrt{e}$

Also $f''(x) = \frac{(x^3)(1 - 2 \ln x) - (1 - 2 \ln x)(x^3)}{(x^3)^2} =$

$$\frac{(x^3)\left(-2\frac{1}{x}\right) - (1 - 2 \ln x)(3x^2)}{x^6} = \frac{-2x^2 - (1 - 2 \ln x)(3x^2)}{x^6} =$$

$$\frac{-2 - (1 - 2 \ln x)(3)}{x^4} = \frac{6 \ln x - 5}{x^4} = 0 \text{ if } \ln x = \frac{5}{6}.$$

Taking exponentials, we get $x = e^{\ln x} = e^{\frac{5}{6}}$.

The “interesting values” are 0 , $e^{\frac{1}{2}}$, and $e^{\frac{5}{6}}$.

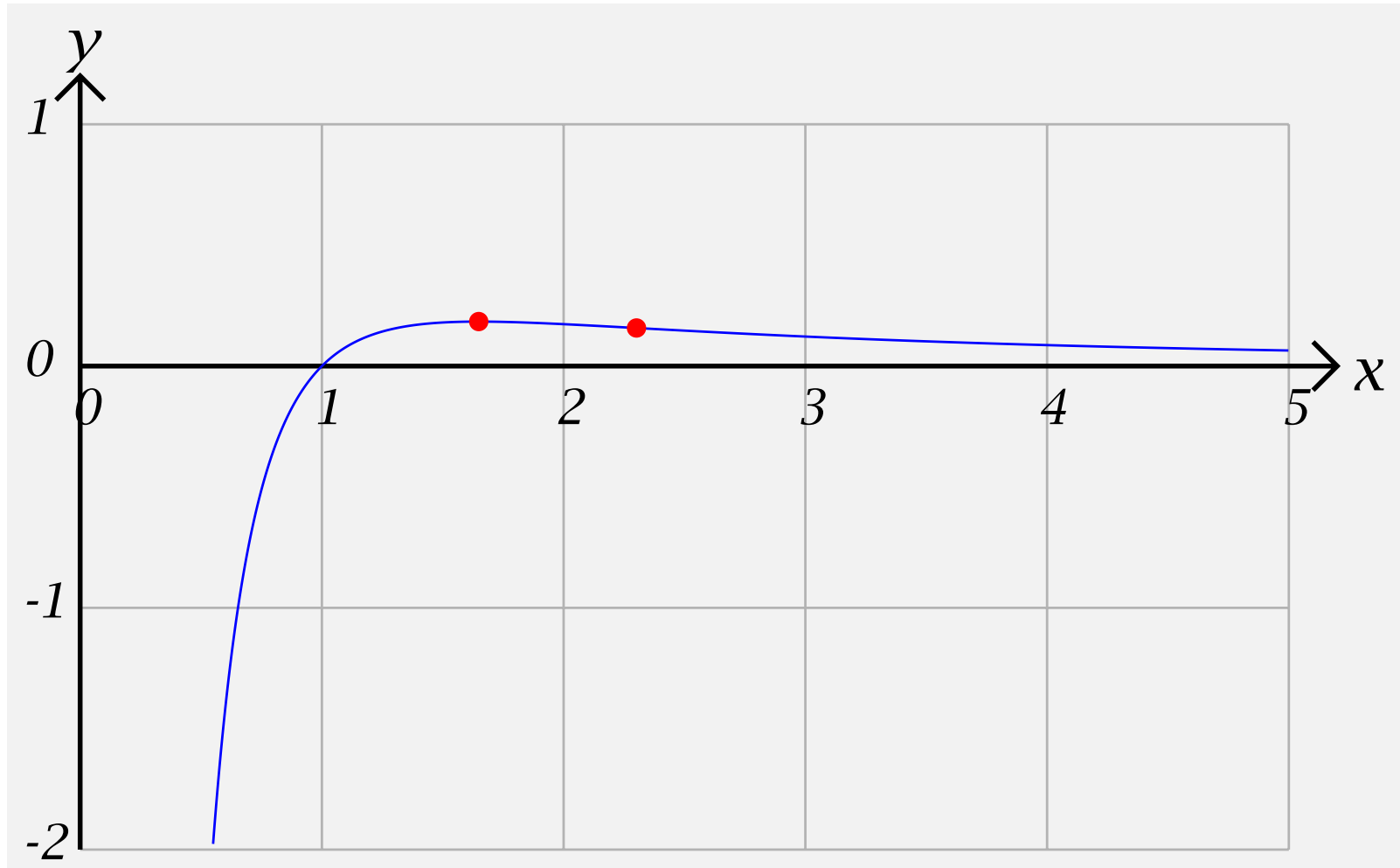
Step 2: Put these values of x into increasing order.

$$0, e^{\frac{1}{2}}, e^{\frac{5}{6}}$$

Step 3: Put together as good a table as you can.

x	$(0, e^{\frac{1}{2}})$	$e^{\frac{1}{2}}$	$(e^{\frac{1}{2}}, e^{\frac{5}{6}})$	$e^{\frac{5}{6}}$	$(e^{\frac{5}{6}}, \infty)$
$f''(x)$	–	–	–	0	+
$f'(x)$	+	0	–	–	–
$f(x)$	–	$\frac{1}{2e} \doteq 1.36$		$\frac{5}{6e^{\frac{5}{3}}} \doteq 0.16$	$f(1) = 0$

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.



Example 10: $f(x) = \frac{\ln x}{x^{\frac{1}{2}}}$

Step 1: The domain of f is $(0, \infty)$.

$$f'(x) = \frac{(x^{\frac{1}{2}})(\ln x) - (\ln x)(x^{\frac{1}{2}})}{(x^{\frac{1}{2}})^2} = \frac{(x^{\frac{1}{2}})\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{x} =$$

$$\frac{\frac{1}{x^{\frac{1}{2}}} - \frac{\frac{1}{2}\ln x}{x^{\frac{1}{2}}}}{x} =$$

$$\frac{2 - \ln x}{2x^{\frac{3}{2}}} = 0 \text{ if } \ln x = 2 \text{ or } x = e^2$$

$$\text{Also } f''(x) = \frac{\left(2x^{\frac{3}{2}}\right)(2 - \ln x) - (2 - \ln x)\left(2x^{\frac{3}{2}}\right)}{\left(2x^{\frac{3}{2}}\right)^2} =$$

$$\frac{\left(2x^{\frac{3}{2}}\right)\left(-\frac{1}{x}\right) - (2 - \ln x)\left(2\frac{3}{2}x^{\frac{1}{2}}\right)}{4x^3} =$$

$$\frac{-2x^{\frac{1}{2}} - (2 - \ln x)\left(3x^{\frac{1}{2}}\right)}{4x^3} = \frac{-2 - 3(2 - \ln x)}{4x^{\frac{5}{2}}} = \frac{3 \ln x - 8}{4x^{\frac{5}{2}}} = 0 \text{ if}$$

$$\ln x = \frac{8}{3} \text{ or } x = e^{\frac{8}{3}}$$

The “interesting values” are 0, e^2 , and $e^{\frac{8}{3}}$.

Step 2: Put these values of x into increasing order.

$$0, e^2, e^{\frac{8}{3}}$$

Step 3: Put together as good a table as you can.

x	$(0, e^2)$	e^2	$(e^2, e^{\frac{8}{3}})$	$e^{\frac{8}{3}}$	$(e^{\frac{8}{3}}, \infty)$
$f''(x)$	-	-	-	0	+
$f'(x)$	+	0	-	-	-
$f(x)$	$f(1) = 0$	$\frac{2}{e} \doteq 0.74$		$\frac{8}{3e^{\frac{4}{3}}} \doteq 0.70$	

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.

