Sketching Graphs of Functions

Increasing and Decreasing Functions

Definition: A function f is increasing on an interval (a, b) if $f(x) \leq f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.

If f' exists on the interval, we have $f'(x) \geq 0$ for all x in (a, b).

Definition: A function f is strictly increasing on an interval (a, b) if $f(x) < f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.
If f' exists on the interval, we have $f'(x) > 0$ for all x in (a, b).

Definition: A function f is decreasing on an interval (a, b) if $f(x) \geq f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.

If f' is exists on the interval, we have $f'(x) \leq 0$ for all x in (a, b).

Definition: A function f is strictly decreasing on an interval (a, b) if $f(x) > f(y)$ for every pair of numbers x and y in the interval satisfying $x < y$.
If f' exists on the interval, we have $f'(x) < 0$ for all x in (a, b).

Second Derivatives

The derivative of the derivative of a function f is called its second derivative. We write $f''(x) = (f'(x))'$ or

$$y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$$

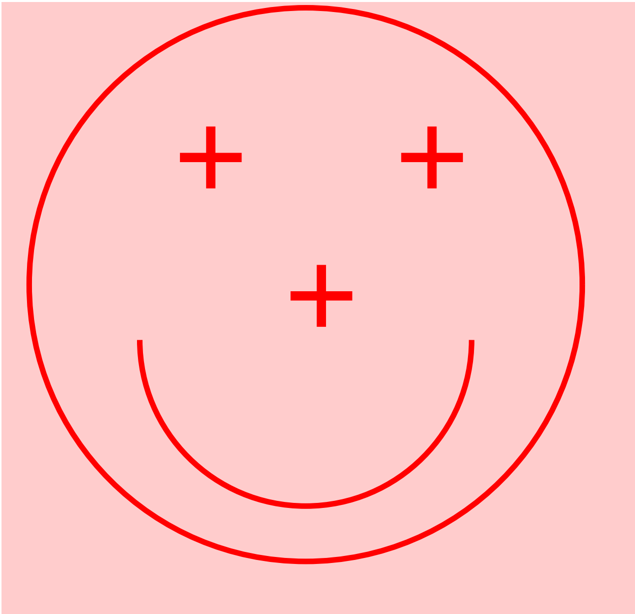
Example: If $f(x) = x^3 + x$ then

$f'(x) = 3x^2 + 1$, and $f''(x) = \left(3x^2 + 1\right)' = 6x$.

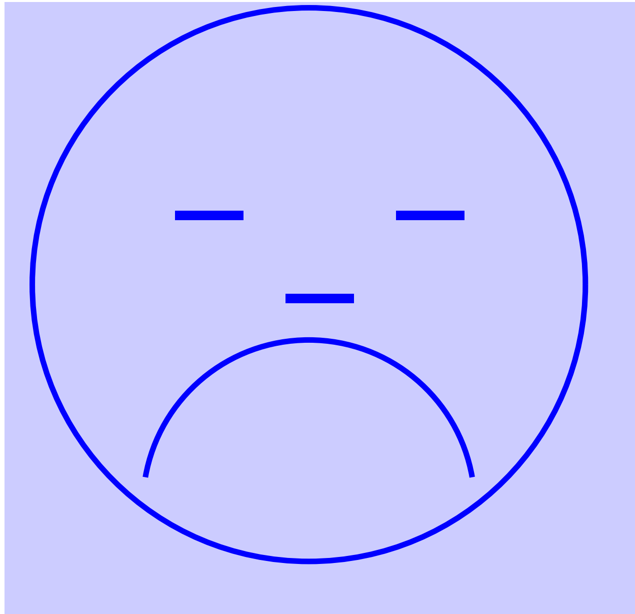
The actual numerical value of $f''(x)$ will not be important in this course, we will only be interested in its sign.

Definition: A function f is concave up on an interval (a, b) if its graph lies above the tangent line to the graph of $y = f(x)$ at every point of the interval.
We have $f''(x) \geq 0$ on such an interval.

Definition: A function f is concave down on an interval (a, b) if its graph lies below the tangent line to the graph of $y = f(x)$ at every point of the interval.
We have $f''(x) \leq 0$ on such an interval.



$f''(x) \geq 0$



$f''(x) \leq 0$

Definition: Any value in the domain of a function where the direction of concavity changes is called an **inflection point**. Such points are characterized by the second derivative changing sign at them.

It is good policy to label all extrema and inflection points when sketching the graph of a function.

An Organized Approach to Sketching the Graph of $y = f(x)$:

Step 1: Compute $f'(x)$ and $f''(x)$ and then find all of the "interesting values" of f: all values of x for which $f'(x)$ or $f''(x)$ equal 0 or are undefined.

Step 2: Put these values of x into increasing order.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.