Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \dfrac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f, and the roots of the denominator give us the possible locations of the vertical asymptotes of f. From the Quotient Rule $f'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f, and the denominator of the second derivative is the fourth power of the denominator of f, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \dfrac{x+1}{x-1}$ is:

Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \frac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f, and the roots of the denominator give us the possible locations of the vertical asymptotes of f. From the Quotient Rule $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f, and the denominator of the second derivative is the fourth power of the denominator of f, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \frac{x+1}{x-1}$ is:

Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$.

The only “interesting value” of f is -1.

Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \frac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f, and the roots of the denominator give us the possible locations of the vertical asymptotes of f. From the Quotient Rule $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f, and the denominator of the second derivative is the fourth power of the denominator of f, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \frac{x+1}{x-1}$ is:

Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$.

The only "interesting value" of f is -1.

Step 2:

Put these values of x into increasing order.

Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \dfrac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f, and the roots of the denominator give us the possible locations of the vertical asymptotes of f. From the Quotient Rule $f'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f, and the denominator of the second derivative is the fourth power of the denominator of f, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \dfrac{x+1}{x-1}$ is:

Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

and $f''(x) = (-2)(-2)(x-1)^{-3} = \dfrac{4}{(x-1)^3}$.

The only "interesting value" of f is -1.

Step 2:

Put these values of x into increasing order. -1.

Sketching Rational Functions

Recall that a rational function $f(x)$ is the quotient of two polynomials — $f(x) = \frac{p(x)}{q(x)}$. Things would be simpler if we could assume that p and q had no common roots, but we cannot. The roots of the numerator $p(x)$ are the possible roots of f, and the roots of the denominator give us the possible locations of the vertical asymptotes of f. From the Quotient Rule $f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}$ we know that the denominator of the first derivative is the square of the denominator of f, and the denominator of the second derivative is the fourth power of the denominator of f, so they contain no new information: The asymptotes are at the roots of $q(x)$ only. The additional information about critical and inflection points comes from the numerators of the first and second derivatives respectively.

Example 1: The graph of $f(x) = \frac{x+1}{x-1}$ is:

Step 1:

$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$-\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

and $f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$.

The only "interesting value" of f is -1.

Step 2:

Put these values of x into increasing order. -1.

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$(-\infty, 1)$	1	$(1, \infty)$
$f''(x)$	$-$	UND	$+$
$f'(x)$	$-$	UND	$-$
$f(x)$		UND	

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

But now we have a problem: the only interesting x-value, 1, doesn't have a corresponding y-value because 1 is not in the domain of f!

Even though the function is not defined at 1, we can compute the left- and right- hand limits there:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x-1} = -\infty, \text{ and } \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} \frac{x+1}{x-1} = +\infty.$$

We can also compute the limits at $-\infty$ and $+\infty$:

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x+1}{x-1} = 1, \text{ and } \lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{x+1}{x-1} = 1.$$

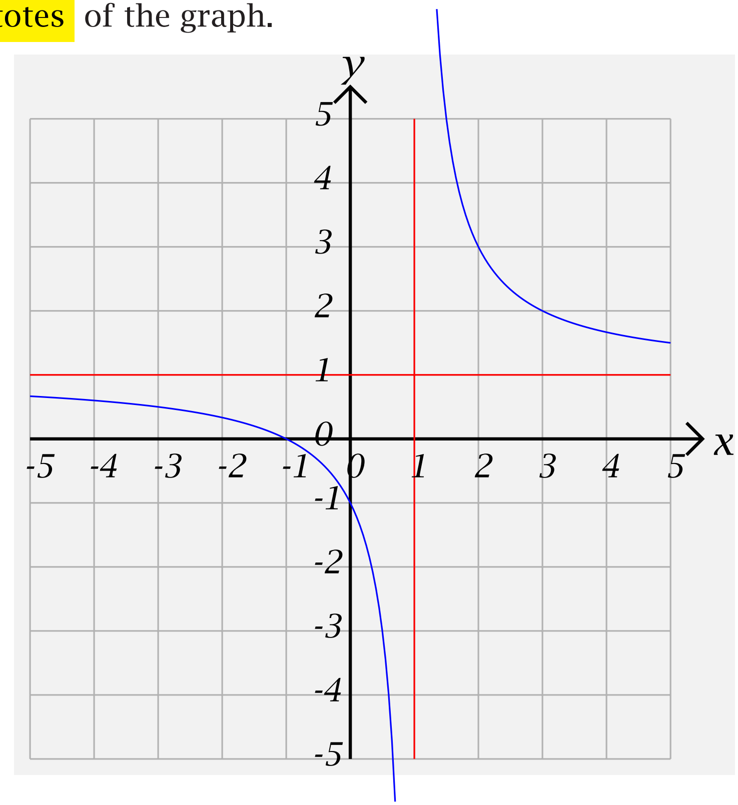
We can also easily see that $f(-1) = 0$. We can represent these facts in a modified table, using a bit of shorthand notation:

x	$-\infty$	$(-\infty, 1)$	1^-	1^+	$(1, \infty)$	∞
$f''(x)$	0	$-$	$-\infty$	$+\infty$	$+$	0
$f'(x)$	0	$-$	$-\infty$	$-\infty$	$-$	0
$f(x)$	1	$f(-1) = 0$	$-\infty$	∞		1

We use this to put together a sketch of the graph: The horizontal line $y = 1$ and the vertical line $x = 1$ are called

x	$-\infty$	$(-\infty, 1)$	1^-	1^+	$(1, \infty)$	∞
$f''(x)$	0	$-$	$-\infty$	$+\infty$	$+$	0
$f'(x)$	0	$-$	$-\infty$	$-\infty$	$-$	0
$f(x)$	1	$f(-1) = 0$	$-\infty$	∞		1

We use this to put together a sketch of the graph: The horizontal line $y = 1$ and the vertical line $x = 1$ are called asymptotes of the graph.



Example 2: The graph of $f(x) = \dfrac{x}{x^2 - 1}$ is:

Example 2: The graph of $f(x) = \dfrac{x}{x^2-1}$ is:

Step 1:

$$f'(x) = \frac{(x^2-1)(x)' - (x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

and $f''(x) = -\dfrac{(x^2-1)^2(x^2+1)' - (x^2+1)\left((x^2-1)^2\right)'}{\left((x^2-1)^2\right)^2} =$

$$-\frac{(x^2-1)^2(2x) - (x^2+1)\left(2(x^2-1)(2x)\right)}{(x^2-1)^4} = -2x\frac{(x^2-1) - (x^2+1)(2)}{(x^2-1)^3} =$$

$$-2x\frac{x^2-1-2x^2-2}{(x^2-1)^3} = -2x\frac{-3-x^2}{(x^2-1)^3} = 2x\frac{3+x^2}{(x^2-1)^3}$$

The only “interesting values” of f are -1, 0, and 1.

Example 2: The graph of $f(x) = \dfrac{x}{x^2 - 1}$ is:

Step 1:

$$f'(x) = \frac{(x^2-1)(x)' - (x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

and $f''(x) = -\dfrac{(x^2-1)^2(x^2+1)' - (x^2+1)\left((x^2-1)^2\right)'}{\left((x^2-1)^2\right)^2} =$

$$-\frac{(x^2-1)^2(2x) - (x^2+1)\left(2(x^2-1)(2x)\right)}{(x^2-1)^4} = -2x\frac{(x^2-1) - (x^2+1)(2)}{(x^2-1)^3} =$$

$$-2x\frac{x^2-1-2x^2-2}{(x^2-1)^3} = -2x\frac{-3-x^2}{(x^2-1)^3} = 2x\frac{3+x^2}{(x^2-1)^3}$$

The only “interesting values” of f are -1, 0, and 1.

Step 2: Put these values of x into increasing order.

Example 2: The graph of $f(x) = \dfrac{x}{x^2 - 1}$ is:

Step 1:

$$f'(x) = \frac{(x^2-1)(x)' - (x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

and $f''(x) = -\dfrac{(x^2-1)^2(x^2+1)' - (x^2+1)\left((x^2-1)^2\right)'}{\left((x^2-1)^2\right)^2} =$

$$-\frac{(x^2-1)^2(2x) - (x^2+1)\left(2(x^2-1)(2x)\right)}{(x^2-1)^4} = -2x\frac{(x^2-1) - (x^2+1)(2)}{(x^2-1)^3} =$$

$$-2x\frac{x^2-1-2x^2-2}{(x^2-1)^3} = -2x\frac{-3-x^2}{(x^2-1)^3} = 2x\frac{3+x^2}{(x^2-1)^3}$$

The only “interesting values” of f are -1, 0, and 1.

Step 2:

Put these values of x into increasing order. $-1, 0, 1.$

Example 2: The graph of $f(x) = \dfrac{x}{x^2-1}$ is:

Step 1:

$$f'(x) = \frac{(x^2-1)(x)' - (x)(x^2-1)'}{(x^2-1)^2} = \frac{(x^2-1)(1)-(x)(2x)}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

and $f''(x) = -\dfrac{(x^2-1)^2(x^2+1)' - (x^2+1)\left((x^2-1)^2\right)'}{\left((x^2-1)^2\right)^2} =$

$$-\frac{(x^2-1)^2(2x)-(x^2+1)\left(2(x^2-1)(2x)\right)}{(x^2-1)^4} = -2x\frac{(x^2-1)-(x^2+1)(2)}{(x^2-1)^3} =$$

$$-2x\frac{x^2-1-2x^2-2}{(x^2-1)^3} = -2x\frac{-3-x^2}{(x^2-1)^3} = 2x\frac{3+x^2}{(x^2-1)^3}$$

The only "interesting values" of f are -1, 0, and 1.

Step 2: Put these values of x into increasing order. $-1, 0, 1.$

Step 3: Put together as good a table as you can showing the signs of $f'(x)$ and $f''(x)$ on the intervals into which the interesting values divide the domain of f.

x	$-\infty$	$(-\infty,-1)$	-1^-	-1^+	$(-1,0)$	0	$(0,1)$	1^-1	1^+	$(1,\infty)$	∞
$f''(x)$	0	$-$	$-\infty$	$+\infty$	$+$	0	$-$	$-\infty$	∞	$+$	0
$f'(x)$	0	$-$	$-\infty$	$-\infty$	$-$	-1	$-$	$-\infty$	$-\infty$	$-$	0
$f(x)$	0	$-$	$-\infty$	∞	$+$	0	$-$	$-\infty$	∞	$+$	0

Step 4: Plot the “interesting points” and connect them with curves which are either left or right half-smiles or half-frowns.

We use this to put together a sketch of the graph:

The horizontal line $y = 0$ and the vertical lines $x = -1$ and $x = 1$ are called

Step 4: Plot the "interesting points" and connect them with curves which are either left or right half-smiles or half-frowns.

We use this to put together a sketch of the graph:

The horizontal line $y = 0$ and the vertical lines $x = -1$ and $x = 1$ are called asymptotes of the graph.