Exercises for Sketching Quartics

Sketch the graphs of:

(1) $y = f(x) = x^4 - 2x^3 - x^2 + 2x.$ Solution

(2) $y = f(x) = x^4 + x^3 - x - 2.$ Solution

(3) $y = f(x) = x^4 - 8x^2 + 16.$ Solution

(4) $y = f(x) = x^4 - 6x^2 - 8x - 3.$ Solution

(5) $y = f(x) = x^4 - 8x^3 - 22x^2 - 24x + 9.$ Solution

(6) $y = f(x) = x^4 + 2x^3 - 12x^2 - 10x + 3.$ Solution

(7) $y = f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12.$ Solution

(8) $y = f(x) = x^4 - 8x^3 + 9x^2 + 8x - 10.$ Solution

Solutions

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(1) $f(x) = x^4 - 2x^3 - x^2 + 2x,$

$f'(x) = 4x^3 - 6x^2 - 2x + 2 = 2(2x^3 - 3x^2 - x + 1)$, $f''(x) = 12x^2 - 12x - 2 = 0$ if

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(12)(-2)}}{2(12)} = \frac{1}{2} \pm \frac{\sqrt{85}}{12}$$

$f'\left(\frac{1}{2}\right) = 0$, so $2x - 1$ is a factor of $f'(x) = 2(2x-1)(x^2 - 2x - 1)$

$x^2 + 2x - 1 = 0$ if $x = \dfrac{2 \pm \sqrt{8}}{2(2)} = \dfrac{1 \pm \sqrt{2}}{2}$

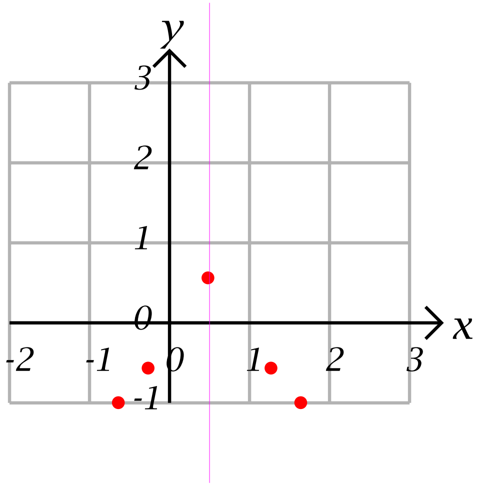
We make a table using interesting values of $x \geq \frac{1}{2}$:

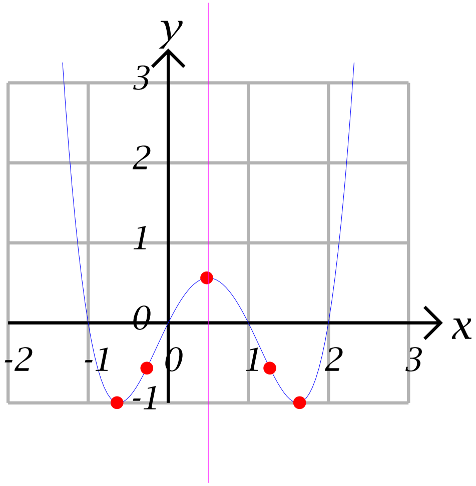
	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{85}}{12}\right)$	$\frac{1}{2} + \frac{\sqrt{85}}{12}$	$\left(\frac{1}{2} + \frac{\sqrt{85}}{12}, \frac{1}{2} + \frac{\sqrt{2}}{2}\right)$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	$\left(\frac{1}{2} + \frac{\sqrt{2}}{2}, \infty\right)$	∞
$f''(x)$	$-$	$-$	0	$+$	$+$	$+$	∞
$f'(x)$	0	$-$	$-$	$-$	0	$+$	∞
$f(x)$	$\frac{9}{16}$	?	$-\frac{11,711}{20,736}$	$-$	$-\frac{9}{16}$	?	∞

Since this graph appears to be symmetric about the line $x = \frac{1}{2}$, we shift the function to the left by $\frac{1}{2}$:

Let $g(x) = f\left(x + \frac{1}{2}\right) = \left(x + \frac{1}{2}\right)^4 - 2\left(x + \frac{1}{2}\right)^3 - \left(x + \frac{1}{2}\right)^2 + 2\left(x + \frac{1}{2}\right) =$

$x^4 - \frac{5}{2}x^2 + \frac{9}{16}$, an even function, so the graph of g is symmetric about the y-axis, and therefore the graph of f is symmetric about the line $x = \frac{1}{2}$





(2) $f(x) = x^4 + x^3 - x - 2,$

$f'(x) = 4x^3 + 3x^2 - 1$, $f''(x) = 12x^2 + 6x = 6x(2x+1)0$ if $x = -\frac{1}{2}$ or 0.

$f'\left(-\frac{1}{2}\right) = -\frac{3}{4}$, and $f'(0) = -1$, so $f'(x)$ has exactly one root. Since $f'(1) = 6$ this root is in the interval $(0, 1)$.
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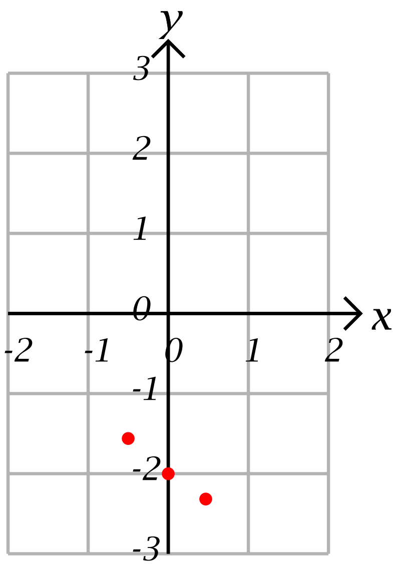
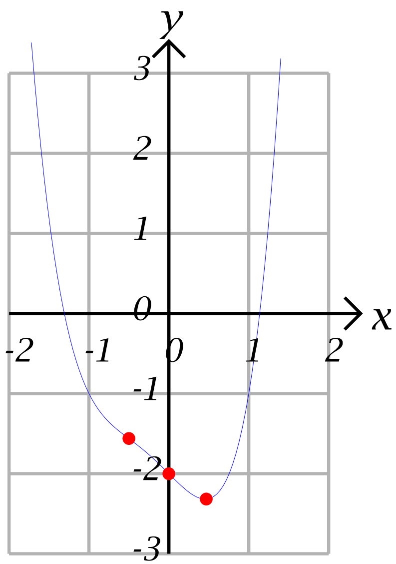
We will call it α. (It is approximately 0.47.)

We make a table using interesting values of x:

	$-\infty$	$\left(-\infty, -\frac{1}{2}\right)$	$-\frac{1}{2}$	$\left(-\frac{1}{2}, 0\right)$	0	$(0, \alpha)$	α	(α, ∞)	∞
$f''(x)$	+	+	0	−	0	+	+	+	∞
$f'(x)$	−	−	−	−	−	−	0	+	∞
$f(x)$	∞	?	$-\frac{25}{16}$	−	−2	−	−	?	∞

We have used the fact that $f(1) = -1$ to conclude that $f(\alpha) < 0$.





(3) $f(x) = x^4 - 8x^2 + 16,$

$$f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2),\ f''(x) = 12x^2 - 16 = 12\left(x^2 - \frac{3}{4}\right) = 0 \text{ if } x = \pm\frac{\sqrt{3}}{2}.$$

Since f is even, we need only build our table over the interval $[0, \infty)$:

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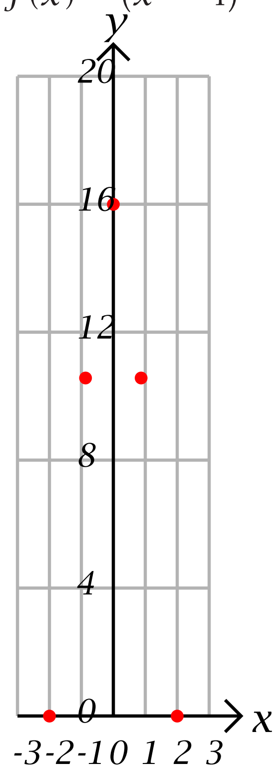
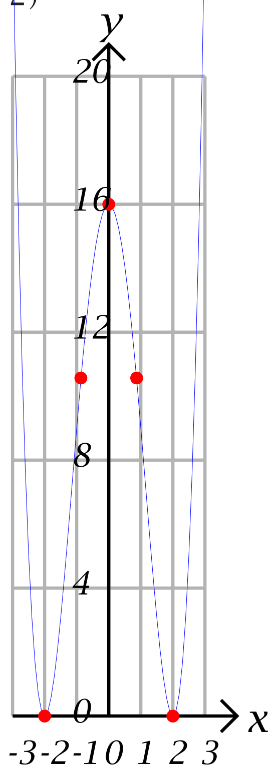
	0	$\left(0, \frac{\sqrt{3}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\left(\frac{\sqrt{3}}{2}, 2\right)$	2	$(2, \infty)$	∞
$f''(x)$	$-$	$-$	0	$+$	$+$	$+$	∞
$f'(x)$	0	$-$	$-$	$-$	0	$+$	∞
$f(x)$	16	$+$	$\frac{169}{16}$	$+$	0	$+$	∞

Note that $f(x) = (x^2-4)^2 = (x-2)^2(x+2)^2$





(4) $f(x) = x^4 - 6x^2 - 8x - 3$,

$f'(x) = 4x^3 - 12x - 8 = 4(x^3 - 3x - 2) = 4(x-2)(x+1)^2$,
$f''(x) = 12x^2 - 12 = 12\left(x^2 - 1\right) = 12(x-1)(x+1) = 0$ if $x = \pm 1$. The interesting points are thus -1, 1, and 2. Note that $f(x) = (x+1)^3(x-3)^2$.

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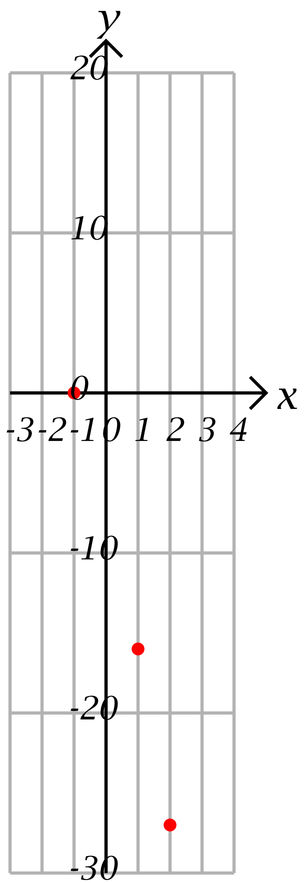
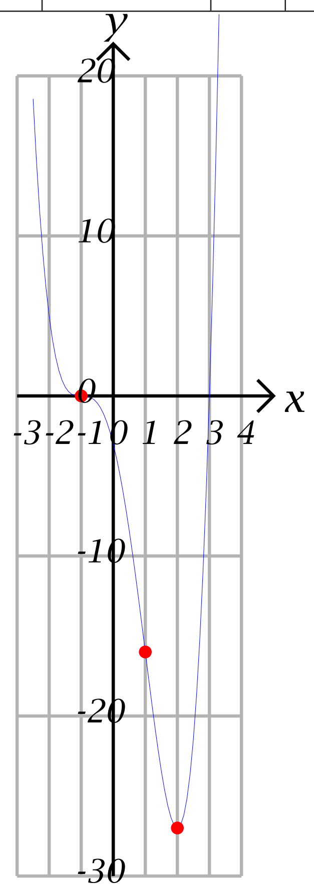
We construct our table:

	$-\infty$	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,2)$	2	$(2,\infty)$	∞
$f''(x)$	$+$	$+$	0	$-$	0	$+$	$+$	$+$	∞
$f'(x)$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	∞
$f(x)$	∞	$+$	0	$-$	-16	$-$	-27	?	∞





(5) $y = f(x) = x^4 - 8x^3 + 22x^2 - 24x + 9 = (x-1)^2(x-3)^2$,

$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$, $f''(x) = 12x^2 - 48x + 44 = 4\left(3x^2 - 12x + 11\right) = 0$

if $x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)} = \dfrac{12 \pm \sqrt{12}}{6} = \dfrac{12 \pm 2\sqrt{3}}{6} = \dfrac{6 \pm \sqrt{3}}{3} = 2 \pm \dfrac{\sqrt{3}}{3}$.

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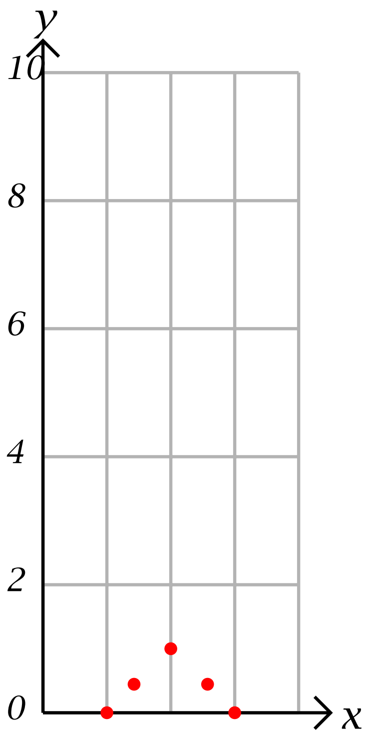
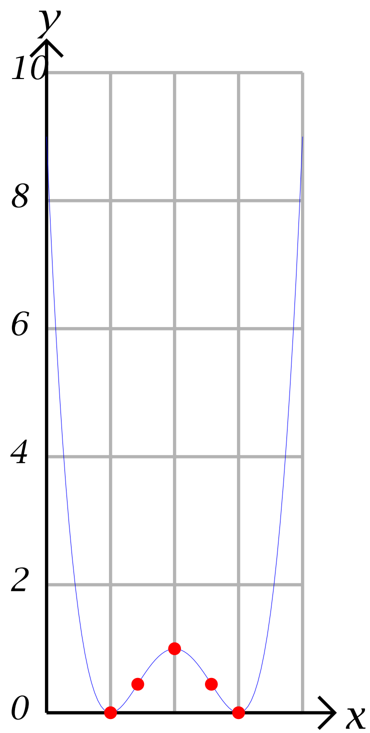
The interesting points are thus 1, $2 - \dfrac{\sqrt{3}}{3}$, $2 + \dfrac{\sqrt{3}}{3}$ and 3.

We construct our table:

	$-\infty$	$(-\infty, 1)$	1	$\left(1, 2 - \frac{\sqrt{3}}{3}\right)$	$2 - \frac{\sqrt{3}}{3}$	$\left(2 - \frac{\sqrt{3}}{3}, 2\right)$	2	$\left(2, 2 + \frac{\sqrt{3}}{3}\right)$	$2 + \frac{\sqrt{3}}{3}$	$\left(2 + \frac{\sqrt{3}}{3}, 3\right)$	3	$(3, \infty)$	∞
$f''(x)$	$-\infty$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	∞
$f'(x)$	$-\infty$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	∞
$f(x)$	∞	$+$	0	$+$	$\frac{1}{9}$	$+$	1	$+$	$\frac{1}{9}$	$+$	0	$+$	∞





(6) $y = f(x) = x^4 + 2x^3 - 12x^2 - 10x + 3 = (x-3)(x+1)\left(x - (-2-\sqrt{5})\right)\left(x - (-2+\sqrt{5}\right)$,
$f'(x) = 4x^3 + 6x^2 - 24x - 10 = 4(x-2)(x-?)(x-?)$, $f''(x) = 12x^2 + 12x - 24 = 12(x-1)(x+2) = 0$ if $x = -2, 1$.

The interesting numbers that we can find are thus -2, and 1. We cannot easily find the roots of $f'(x)$, so we construct a smaller table, containing only the inflection numbers:

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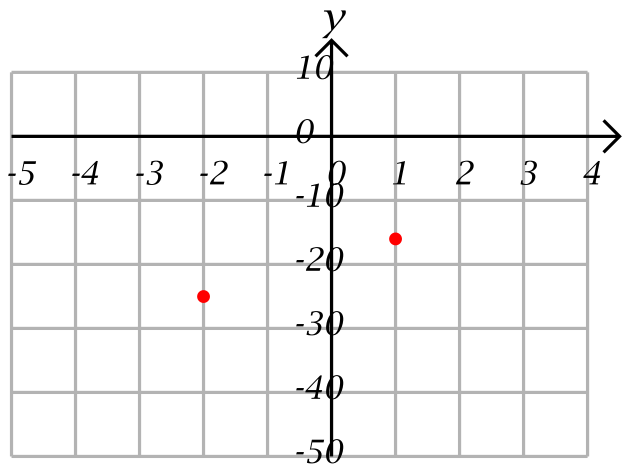
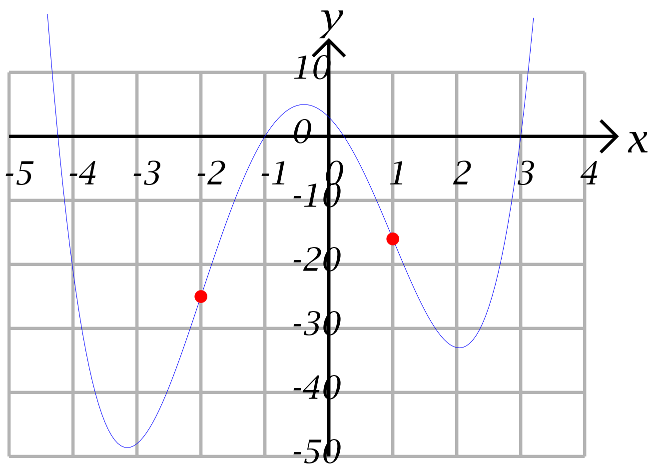
	$-\infty$	$(-\infty,-2)$	-2	$(-2,1)$	1	$(1,\infty)$	∞
$f''(x)$	$-\infty$	$+$	0	$-$	0	$+$	∞
$f'(x)$	$-\infty$	$-$	30	$?$	-24	$?$	∞
$f(x)$	∞	$+$	-25	$f(-1)=0$	-16	$f(3)=0$	∞





(7) $y = f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12 =$,

$f'(x) = 4x^3 - 6x^2 - 14x + 8 = 2(2x^3 - 3x^2 - 7x + 4) = 2(2x-1)(x^2 - x - 4)$, which has roots $\frac{1}{2}$ and $\frac{1 \pm \sqrt{17}}{2}$, and

$f''(x) = 12x^2 - 12x - 14 = 2(6x^2 - 6x - 7) = 0$ if $x = \frac{1}{2} \pm \frac{\sqrt{30}}{6}$.

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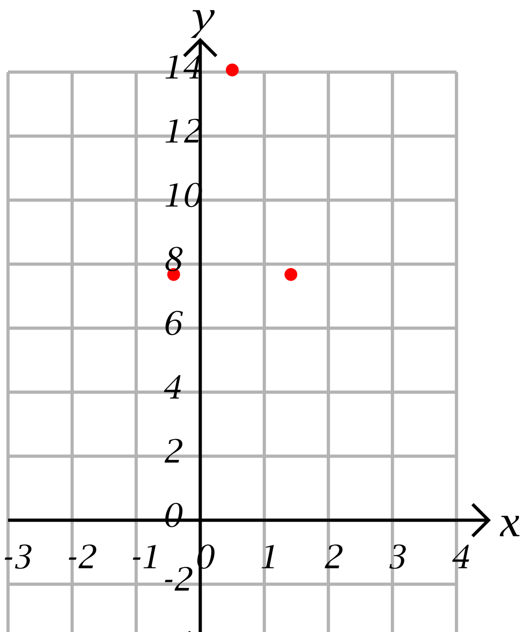
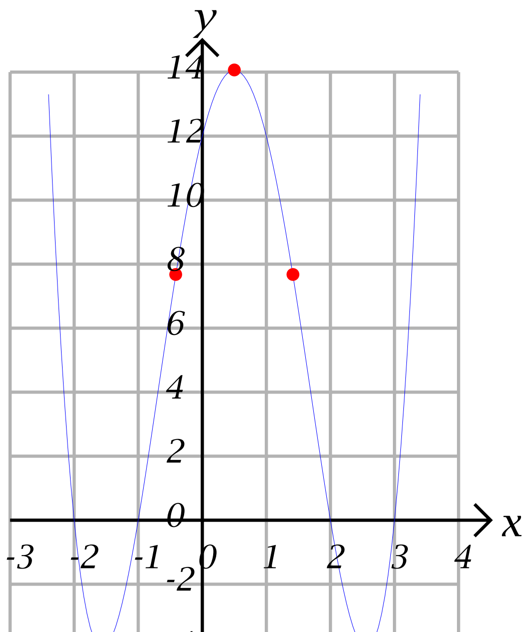
The interesting numbers are thus $\frac{1}{2}, \frac{1 \pm \sqrt{17}}{2}, \frac{1}{2} \pm \frac{\sqrt{30}}{6}$. Since f appears to be symmetric about the line $x = \frac{1}{2}$, we verify that $f\left(\frac{1}{2} + x\right) = f\left(\frac{1}{2} - x\right)$ and decide to only look at its values $\geq x = \frac{1}{2}$. We construct a table:

	$\frac{1}{2}$	$\left(\frac{1}{2}, \frac{1}{2} + \frac{\sqrt{30}}{6}\right)$	$\frac{1}{2} + \frac{\sqrt{30}}{6}$	$\left(\frac{1}{2} + \frac{\sqrt{30}}{6}, \frac{1}{2} + \frac{\sqrt{17}}{2}\right)$	$\frac{1}{2} + \frac{\sqrt{17}}{2}$	$\left(\frac{1}{2} + \frac{\sqrt{17}}{2}, \infty\right)$	∞
$f''(x)$	$-$	$-$	0	$+$	$+$	$+$	∞
$f'(x)$	0	$-$	$-$	$-$	0	$+$	∞
$f(x)$	$\frac{225}{16}$	$f(2) = 0$	$-$	$-$	-4	$f(3) = 0$	∞





(8) $y = f(x) = x^4 - 8x^3 + 9x^2 + 8x - 10 = (x^2 - 1)(x^2 - 8x + 10)$.

$f'(x) = 4x^3 - 24x^2 + 18x + 8 = 2(2x^3 - 12x^2 + 9x + 4)$, whose roots are not easy to find, and involve cube roots.

$f''(x) = 12x^2 - 48x + 18 = 6(2x^2 - 8x + 3) = 0$ if $x = 2 \pm \frac{\sqrt{10}}{2}$. The inflection numbers are thus $2 - \frac{\sqrt{10}}{2}$ and $2 + \frac{\sqrt{10}}{2}$. We construct a table:

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	$-\infty$	$\left(-\infty, 2 - \frac{\sqrt{10}}{2}\right)$	$2 - \frac{\sqrt{10}}{2}$	$\left(2 - \frac{\sqrt{10}}{2}, 2 + \frac{\sqrt{10}}{2}\right)$	$2 + \frac{\sqrt{10}}{2}$	$\left(2 + \frac{\sqrt{10}}{2}, \infty\right)$	∞
$f''(x)$	∞	$+$	0	$-$	0	$+$	∞
$f'(x)$	$-\infty$	$f'(0) = 8$	?	$f'(1) = 2, f'(2) = -20$	?	?	∞
$f(x)$	∞	$f(-1) = 0$	?	$f(1) = 0$		?	∞