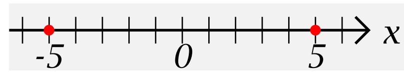
Absolute Value

The **absolute value** $|x|$ of a number x is defined to be its distance from the number 0.

Thus $|5| = 5$, and $|-5| = 5$. Note that $-|-5| = -5$.



We have two general formulas for x:

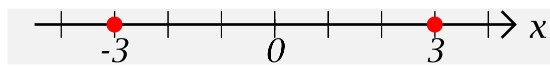
$$|x| = \sqrt{x^2} \qquad \text{and} \qquad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, if we wish to find $|5|$ we may either use $|5| = \sqrt{5^2} = \sqrt{25} = 5$, or we may observe that $|5| = 5$ since $5 \geq 0$. This is easy when we know the value of x in $|x|$, but becomes less obvious when x is unknown.

Example 1: "Solve the equation $|x| = 3$"

Geometric Solution: Mark off the two points on the number line that are 3 units from 0:

The solution set is $\{-3, 3\}$

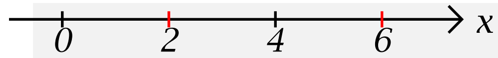


Algebraic Solution 1: We know that either $|x| = x$ or $|x| = -x$, so we solve the two equations $x = 3$ and $-x = 3$ to get the solution set $\{-3, 3\}$

Algebraic Solution 2: We know that $|x| = \sqrt{x^2} = 3$ so $x^2 = 3^2 = 9$ is a quadratic equation with solutions $x = 3$ and $x = -3$ so we again get the solution set $\{-3, 3\}$

Note: $|a-b| = |b-a|$ is the distance of the number b from the number a.

Example 2: "Solve the equation $|x-4| = 2$"



Geometric Solution: Mark off the two points on the number line that are 2 units from 4. The solution set is $\{2, 6\}$

Algebraic Solution 1: We know that either $|x-4| = x-4$ or $|x-4| = -(x-4) = 4-x$, so we solve the two equations $x-4 = 2$ and $4-x = 2$ to get the solution set $\{2, 6\}$

Algebraic Solution 2: We know that $|x-4| = \sqrt{(x-4)^2} = 2$ so $(x-4)^2 = 2^2 = 4$ is a quadratic equation with solutions $x-4 = -2$ and $x-4 = -2$ so we again get the solution set $\{-3, 3\}$

Properties of Absolute Value

$|x| = |-x|$
$|a-b| = |b-a|$ is the distance of the number b from the number a.

Thus $|a+b| = |a-(-b)| = |-(-a)+b|$ is the distance of the number $-b$ from the number a and the distance of the number $-a$ from the number b.

$|a+b| \leq |a| + |b|$
If a and b have the same sign then we have an equality: $|a+b| = |a| + |b|$, but
If a and b have opposite sign then we certainly have an inequality: $|a+b| < |a| + |b|$.

When in doubt, try simple numerical examples, such as;

$|5+(-3)| = |2| = 2 < |5| + |-3| = 5 + 3 = 8$

$|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (if $b \neq 0$.)

Example 3: "Solve the equation $|4x - 3| = |x + 6|$"

Geometric Solution: This is a bit trickier, but worth doing for the insights we gain: a bit of manipulation is needed:

$|4x - 3| = |x + 6| \iff$

$|4(x - 3/4)| = |x - (-6)| \iff$

$|4||x - 3/4| = |x - (-6)| \iff$

$4|x - 3/4| = |x - (-6)|$

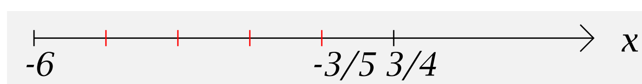
so the distance of x from -6 must be 4 times its distance from $3/4$.

Thus x lies between -6 and $3/4$ or to the right of $3/4$.

We find the value between -6 and $3/4$ by dividing the interval $[-6, 3/4]$ into fifths:

the length of the interval $[-6, 3/4]$ is $\frac{27}{4}$,

so one-fifth of it is $\frac{1}{5}(3/4 - (-6)) = \frac{1}{5}\frac{27}{4} = \frac{27}{20}$,

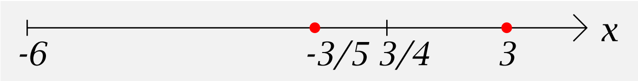


so the desired value of x is

$$\frac{3}{4} - \frac{27}{20} = \frac{5}{5}\frac{3}{4} - \frac{27}{20} = \frac{15}{20} - \frac{27}{20} = \frac{15-27}{20} = -\frac{12}{20} = -\frac{3}{5}$$

To find the point to the right of $3/4$, we notice that its distance from -6 must be 4 times its distance d from $3/4$, so we have $4d = d + \frac{27}{4}$ or $3d = \frac{27}{4}$ or $d = \frac{9}{4}$. Thus $x = \frac{3}{4} + \frac{9}{4} = \frac{12}{4} = 3$.

The solution set is $\left\{-\frac{3}{5}, 3\right\}$



Algebraic Solution 1: We know that either $4x - 3 = x + 6$ or $4x - 3 = -(x + 6) = -x - 6$, so we solve the two equations $4x - 3 = x + 6$ and $4x - 3 = -x - 6$:

$$\begin{array}{lcll}
4x - 3 & = & x + 6 & \iff \\
4x + (-x) - 3 & = & x + (-x) + 6 & \iff \\
3x - 3 & = & 6 & \iff \\
3x - 3 + 3 & = & 6 + 3 & \iff \\
3x & = & 9 & \iff \\
\frac{1}{3}(3x) & = & \frac{1}{3}(9) & \iff \\
x & = & 3 &
\end{array}$$

OR

$$\begin{array}{lcll}
4x - 3 & = & -x - 6 & \iff \\
4x + x - 3 & = & -x + x - 6 & \iff \\
5x - 3 & = & -6 & \iff \\
5x - 3 + 3 & = & -6 + 3 & \iff \\
5x & = & -3 & \iff \\
\frac{1}{5}(5x) & = & \frac{1}{5}(-3) & \iff \\
x & = & -\frac{3}{5} &
\end{array}$$

to get the solution set $\left\{-\frac{3}{5}, 3\right\}$

Algebraic Solution 2: We wish to solve $|4x-3| = |x+6|$, and we know that $|4x-3| = \sqrt{(4x-3)^2}$ and $|x+6| = \sqrt{(x+6)^2}$, so $(4x-3)^2 = (x+6)^2$ is a quadratic equation in x which simplifies as follows:

$16x^2 - 24x + 9 = x^2 + 12x + 36$

$15x^2 - 36x - 27 = 0$

$5x^2 - 12x - 9 = 0$

which has solution $x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(5)(-9)}}{2(5)} = \dfrac{12 \pm \sqrt{144+180}}{10} =$

$\dfrac{12 \pm \sqrt{324}}{10} = \dfrac{12 \pm 2\sqrt{81}}{10} = \dfrac{6 \pm \sqrt{81}}{5} = \dfrac{6 \pm 9}{5} = \dfrac{15}{5}, \dfrac{-3}{5} = 3, -\dfrac{3}{5}$

so we again get the solution set $\left\{-\dfrac{3}{5}, 3\right\}$

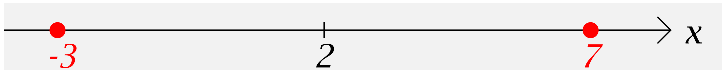
Example 4: Solve the inequalities $|x| < 5$ and $|x| > 5$.

Solution: $(-5, 5)$ and $(-\infty, -5) \cup (5, \infty)$

Example 5: "Solve $|x-2| < 5$"

The solution set consists of all numbers which are less than a distance 5 from 2, so all we need to do is find the two numbers which are exactly a distance 5 from 2, namely 2-5=-3 and 2+5=7.

The solution set is thus $(-3, 7)$.



Example 6: " Solve $|2 - 7x| - 1 > 4$"

First add 1 to both sides to get the equivalent inequality $|2 - 7x| > 5$.

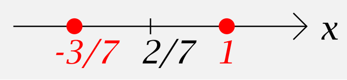
Next, factor 7 out of the left hand side: $|2 - 7x| = |7(\frac{2}{7} - x)| = |7||\frac{2}{7} - x| = 7|\frac{2}{7} - x|$,

so another equivalent inequality is $7|\frac{2}{7} - x| > 5$

Multiplying both sides by $\frac{1}{7}$, we get $\frac{1}{7}7|\frac{2}{7} - x| > \frac{1}{7}5$
or $|\frac{2}{7} - x| > \frac{5}{7}$

Thus the solution set consists of all numbers which are further than $\frac{5}{7}$ from $\frac{2}{7}$.

The two numbers which are exactly that distance from $\frac{2}{7}$ are $\frac{2}{7} - \frac{5}{7} = -\frac{3}{7}$ and $\frac{2}{7} + \frac{5}{7} = 1$.



Thus the solution set is $\left(-\infty, -\frac{3}{7}\right) \cup (1, \infty)$.

Example 7: "Solve $|2 - 5x| \geq -4$."

Always true, solution set is $(-\infty, \infty)$.

Problem:

Express the condition $a \leq x \leq b$ using absolute values and an inequality.

The solution set is $[a, b]$, and we wish to describe this with absolute values and an inequality.

The midpoint of the interval is $\dfrac{a+b}{2}$, and the length of the interval is $b - a$,

so the desired inequality is $\left|x - \dfrac{a+b}{2}\right| \leq \dfrac{b-a}{2}$.

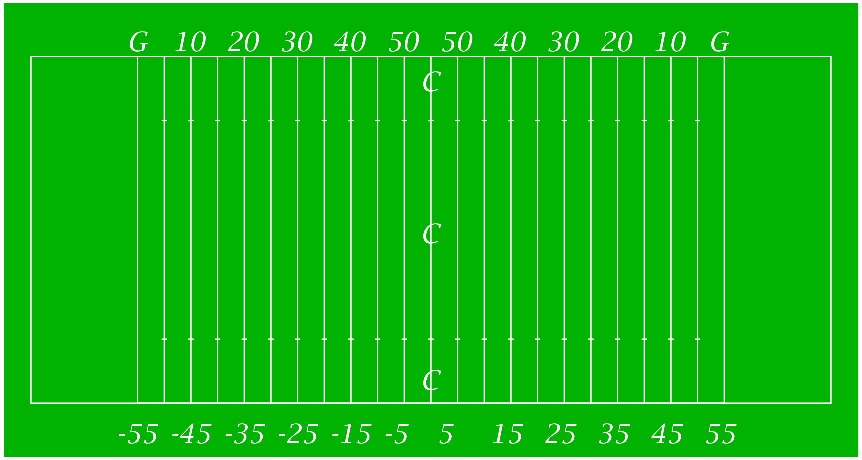
Example 8: Express the condition $32 < x < 212$ using absolute values and an inequality.

The midpoint of the interval (32,212) is 122, and its length is 180, so our solution is

$$|x - 122| < 90$$

Example 9: Express the condition $x < 0$ or $100 < x$ using absolute values and an inequality.

The midpoint of the interval (0,100) is 50 and its length is 100, so our solution is $|x - 50| > 50$



If a football is kicked from the 35-yard line and lands on the opposing 25 yard line, how far has it been kicked? What does this have to do with absolute values?