

Interval Notation

We will often need to describe sets of numbers called **intervals**. Assuming that $a < b$, we call a the **left-hand** endpoint and b the **right-hand** endpoint of the four kinds of **finite** interval (depicted in red):

$(a, b) = \{x | a < x < b\}$, the set of all numbers between a and b , excluding both a and b , usually called the **open** interval from a to b ,



$[a, b] = \{x | a \leq x \leq b\}$, the set of all numbers between a and b , including both a and b , usually called the **closed** interval from a to b ,



$[a, b) = \{x | a \leq x < b\}$, the set of all numbers between a and b , including a but not b .



$(a, b] = \{x | a < x \leq b\}$, the set of all numbers between a and b , including b but not a .



Finite intervals have a **midpoint** whose numerical value is $\frac{a+b}{2}$, the average value of a and b .

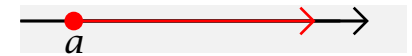
The **length** of a finite interval is $b - a$.

Infinite Intervals

$(a, \infty) = \{x | a < x\}$, the set of all numbers greater than a , not including a .



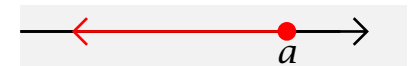
$[a, \infty) = \{x | a \leq x\}$, the set of all numbers greater than or equal to a , including a .



$(-\infty, a) = \{x | x < a\}$, the set of all numbers less than a , not including a .



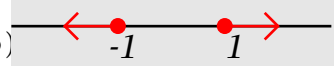
$(-\infty, a] = \{x | x \leq a\}$, the set of all numbers less than or equal to a , including a .



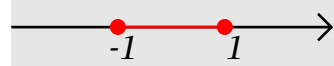
Unions and Intersections of Sets

If A and B are sets, we let $A \cup B$ denote the set containing anything which belongs to either A or B , and we let $A \cap B$ denote the set containing only those things which belong to both A and B .

Example: The set of numbers which satisfy $x \leq -1$ or $1 \leq x$ can be written as $(-\infty, -1] \cup [1, \infty)$



Example: The set of numbers which satisfy $x \leq 1$ and $-1 \leq x$ can be written as $(-\infty, 1] \cap [-1, \infty) = [-1, 1]$



Linear Inequalities: An Introduction

An “inequality” is a statement that two quantities are unequal.

Some examples involving actual numbers are:

$$3 \neq 5, \quad 2 < 4, \quad -3 \leq -2, \quad 5 > 3, \quad \pi \geq 3$$

Notice that there are 5 different inequality symbols possible:

$$\neq, \quad <, \quad \leq, \quad >, \quad \geq$$

Some examples involving variables are:

$$|x| \neq 3, \quad x^2 < 3, \quad x^3 \leq 1, \quad x + 2 > 4, \quad 3x \geq 12$$

An inequality is said to be **linear** if there no powers other than 1 of the variables in it. Which of the above inequalities are linear?

Properties of Inequalities

Addition of a Constant

If \spadesuit is any one of the five possible inequality symbols, we will have: If $a \spadesuit b$, then $a + c \spadesuit b + c$, that is:

If $a \neq b$, then $a + c \neq b + c$

If $a < b$, then $a + c < b + c$

If $a \leq b$, then $a + c \leq b + c$

If $a > b$, then $a + c > b + c$

If $a \geq b$, then $a + c \geq b + c$

Multiplication by a Positive Constant

If $a \spadesuit b$, and c is positive, then $ac \spadesuit bc$, that is:

If $a \neq b$, then $ac \neq bc$

If $a < b$, then $ac < bc$

If $a \leq b$, then $ac \leq bc$

If $a > b$, then $ac > bc$

If $a \geq b$, then $ac \geq bc$

Multiplication by a Negative Constant

If $a \spadesuit b$, and c is negative, then the direction of the inequality changes:

If $a \neq b$, then $ac \neq bc$

If $a < b$, then $ac > bc$

If $a \leq b$, then $ac \geq bc$

If $a > b$, then $ac < bc$

If $a \geq b$, then $ac \leq bc$

Solution Sets

We shall often wish to write the set of numbers that satisfy an inequality in interval notation.

Example 1: “Solve $3x + 5 > 11$ ”

To do this, we can work in two different ways:

<p>Equivalent Inequalities</p> $3x + 5 > 11 \iff$ $3x + 5 + (-5) > 11 + (-5) \iff$ $3x > 6 \iff$ $\frac{1}{3}3x > 6\frac{1}{3} \iff$ $x > 2$ <p>which has solution set $(2, \infty)$.</p>	<p>Set Equations</p> $\{x \mid 3x + 5 > 11\} =$ $\{x \mid 3x + 5 + (-5) > 11 + (-5)\} =$ $\{x \mid 3x > 6\} =$ $\{x \mid \frac{1}{3}3x > 6\frac{1}{3}\} =$ $\{x \mid x > 2\} = (2, \infty).$
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Example 2: “Solve $4 - 3x \leq 7 + 2x$ ”

Again we can work in at least two different ways:

<p>Equivalent Inequalities</p> $4 - 3x \leq 7 + 2x \iff$ $4 - 3x + 3x \leq 7 + 2x + 3x \iff$ $4 \leq 7 + 5x \iff$ $4 + (-7) \leq 7 + (-7) + 5x \iff$ $-3 \leq 5x \iff$ $\frac{1}{5}(-3) \leq \frac{1}{5}5x \iff$ $-\frac{3}{5} \leq x$ <p>which has solution set $[-\frac{3}{5}, \infty)$.</p>	<p>Set Equations</p> $\{x \mid 4 - 3x \leq 7 + 2x\} =$ $\{x \mid 4 - 3x + 3x \leq 7 + 2x + 3x\} =$ $\{x \mid 4 \leq 7 + 5x\} =$ $\{x \mid 4 + (-7) \leq 7 + (-7) + 5x\} =$ $\{x \mid -3 \leq 5x\} =$ $\{x \mid \frac{1}{5}(-3) \leq \frac{1}{5}5x\} =$ $\{x \mid \frac{3}{5} \leq x\} = [-\frac{3}{5}, \infty).$
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or we can vary the order of algebraic operations:

Equivalent Inequalities

$$\begin{aligned}
 4 - 3x &\leq 7 + 2x \iff \\
 4 - 3x + (-2x) &\leq 7 + 2x + (-2x) \iff \\
 4 - 5x &\leq 7 \iff \\
 4 + (-4) - 5x &\leq 7 + (-4) \iff \\
 -5x &\leq 3 \iff \\
 -\frac{1}{5}(-5x) &\geq -\frac{1}{5}3 \iff \\
 x &\geq -\frac{3}{5}
 \end{aligned}$$

which has solution set $\left[-\frac{3}{5}, \infty\right)$.

Set Equations

$$\begin{aligned}
 \{x \mid 4 - 3x &\leq 7 + 2x\} = \\
 \{x \mid 4 - 3x + (-2x) &\leq 7 + 2x + (-2x)\} = \\
 \{x \mid 4 - 5x &\leq 7\} = \\
 \{x \mid 4 + (-4) - 5x &\leq 7 + (-4)\} = \\
 \{x \mid -5x &\leq 3\} = \\
 \left\{x \mid -\frac{1}{5}(-5x) &\geq -\frac{1}{5}3\right\} = \\
 \left\{x \mid x &\geq -\frac{3}{5}\right\} = \left[-\frac{3}{5}, \infty\right).
 \end{aligned}$$



Chained Inequalities

In practical situations it is common to restrict a variable to lying between two values:

For example, we might want to keep an animals daily calory intake, say x , between two levels, such as 1000 and 1500. Mathematically, we would write

$$1000 \leq x \leq 2000$$

by which we mean $1000 \leq x$ AND $x \leq 2000$

A Common Mistake: Mixing Inequality Directions

inequalities and end up with something like

$$1000 \geq x \geq 2000$$

A minute's thought tells that this is ambiguous:

it either means $1000 \geq x$ AND $x \geq 2000$, which is impossible, so the solution set would be empty, or it means $1000 \geq x$ OR $x \geq 2000$,

whose solution set is $(-\infty, 1000] \cup [2000, \infty)$

The mathematical convention is that when we write $a < b < c$ we mean $a < b$ AND $b < c$.

We **ALWAYS** ensure that our chained inequalities run in the same direction, and we **NEVER** use \neq in them.

If we are tempted to write something like

$$1000 \geq x \geq 2000$$

we should realize that it has two different possible meanings, and we should write down the one we really mean.

Correctly Chained Inequalities:

$$a < b < c$$

$$a \leq b < c$$

$$a < b \leq c$$

$$a \leq b \leq c$$

$$a > b > c$$


$$a \geq b > c$$

$$a > b \geq c$$

$$a \geq b \geq c$$

Example 3: “Solve $-2 < 5 + 3x < 20$ ”


Here we are able to work with two inequalities simultaneously, and again we can work in at least two different ways:

Equivalent Inequalities	Set Equations	
$-2 < 5 + 3x < 20 \iff$	$\{x \mid -2 < 5 + 3x < 20\} =$	
$-2 + (-5) < 5 + (-5) + 3x < 20 + (-5) \iff$	$\{x \mid -2 + (-5) < 5 + (-5) + 3x < 20 + (-5)\} =$	
$-7 < 3x < 15 \iff$	$\{x \mid -7 < 3x < 15\} =$	
$\frac{1}{3}(-7) < \frac{1}{3}(3x) < \frac{1}{3}(15) \iff$	$\{x \mid \frac{1}{3}(-7) < \frac{1}{3}(3x) < \frac{1}{3}(15)\} =$	
$-\frac{7}{3} < x < 5 \iff$	$\{x \mid -\frac{7}{3} < x < 5\} = \left(-\frac{7}{3}, 5\right) .$	

which has solution set $\left(-\frac{7}{3}, 5\right)$.

Example 4: “Solve $3x - 5 < 7$ OR $2x - 1 > 13$ ”

Solution by Equivalent Inequalities

$3x - 5 < 7 \iff$	OR	$2x - 1 > 13 \iff$		
$3x - 5 + 5 < 7 + 5 \iff$	OR	$2x - 1 + 1 > 13 + 1 \iff$		
$3x < 12 \iff$	OR	$2x > 14 \iff$		
$\frac{1}{3}(3x) < \frac{1}{3}(12) \iff$	OR	$\frac{1}{2}(2x) > \frac{1}{2}(14) \iff$		
$x < 4$	OR	$x > 7$		

which has solution set $(-\infty, 4) \cup (7, \infty)$.