Coordinate Geometry & Lines

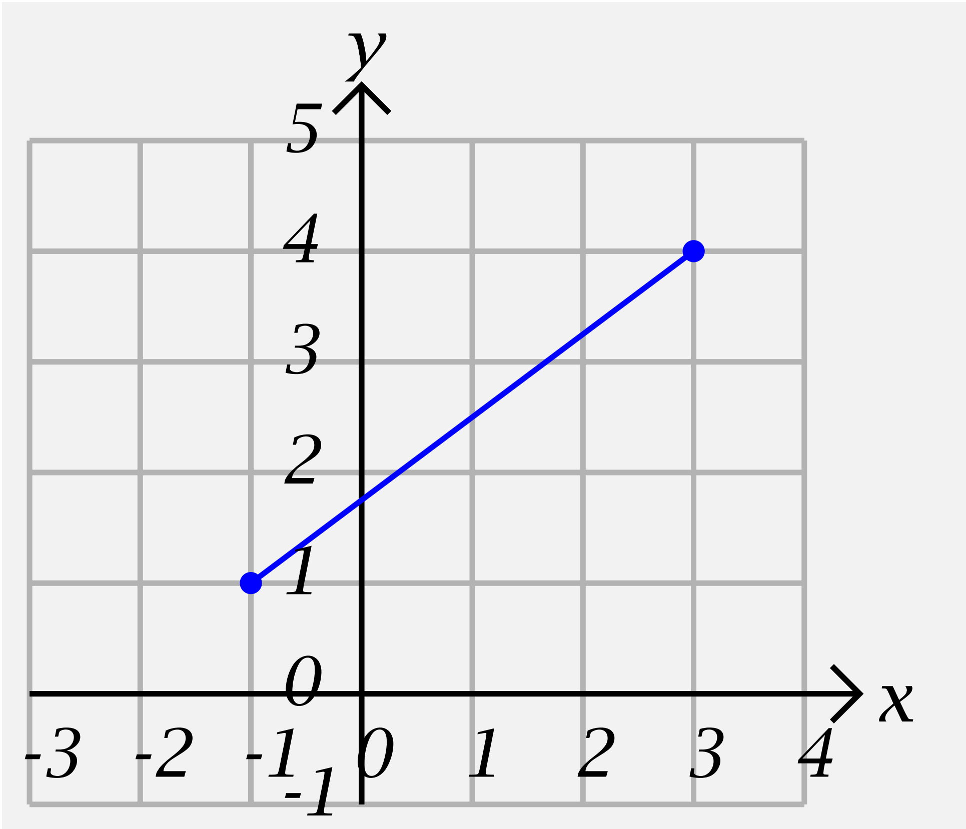
It is essential that the student be able to automatically apply the very basic formulas of elementary Analytic Geometry:

Distance Formula

The distance between the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In our diagram, we take $P_1 = (x_1, y_1)$ to be $(-1, 1)$ and $P_2 = (x_2, y_2)$ to be $(3, 4)$.

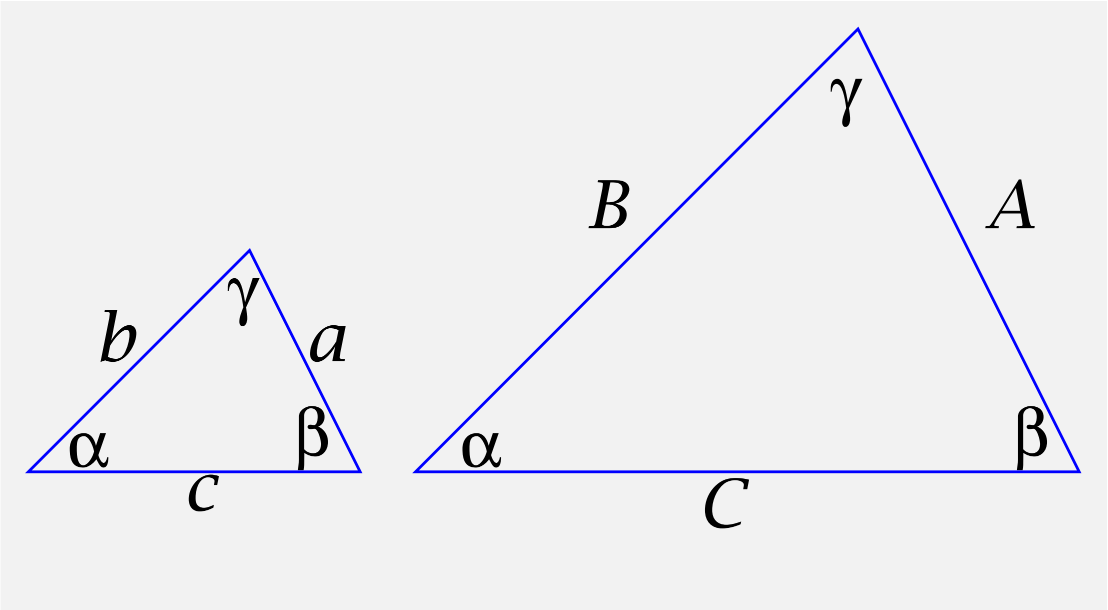


Then $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} =$$

$$\sqrt{(3 - (-1))^2 + (4 - 1)^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

Similar Triangles

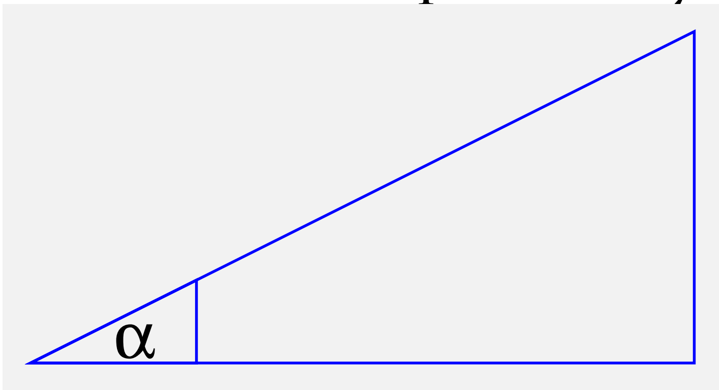
Recall that two triangles are said to be similar if they have identical matching angles.



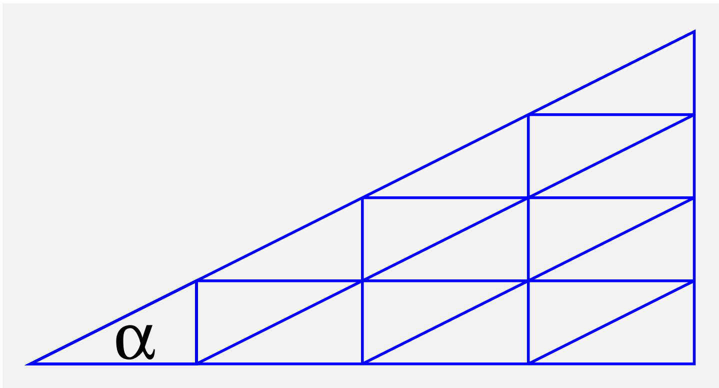
The ratios of the corresponding sides are all equal:

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

In Calculus we will mostly deal with right-angled triangles having a horizontal and a vertical side. The ratio of the vertical side to the horizontal side depends only on the angle α.



If the sides of the similar triangles are integer multiples of each other, it is easy to cut up the bigger triangle into copies of the smaller triangle:



Slope

The slope of the line passing through the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

so in our first example $m = \dfrac{4 - 1}{3 - (-1)} = \dfrac{3}{4}$

Equations of Lines

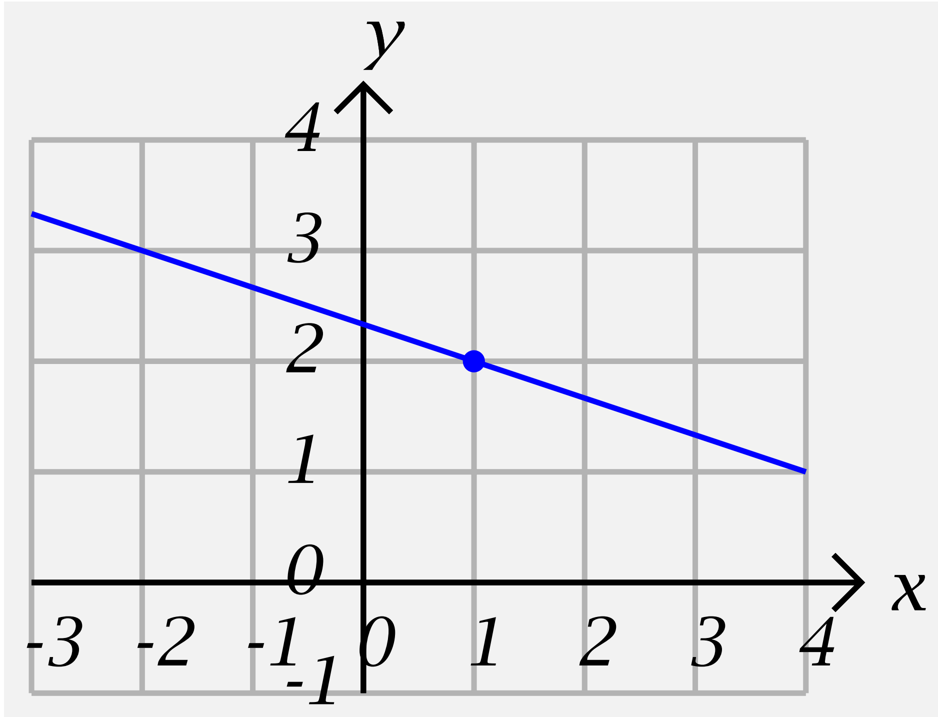
These come in many useful forms:

Point-Slope Form

The equation of the line passing through the point $P_1 = (x_1, y_1)$ with slope m is $y - y_1 = m(x - x_1)$

Thus given the point $P_1 = (1, 2)$ and the slope $m = -\frac{1}{3}$ the equation of the line is

$y - 2 = -\frac{1}{3}(x - 1)$



Point-Point Form

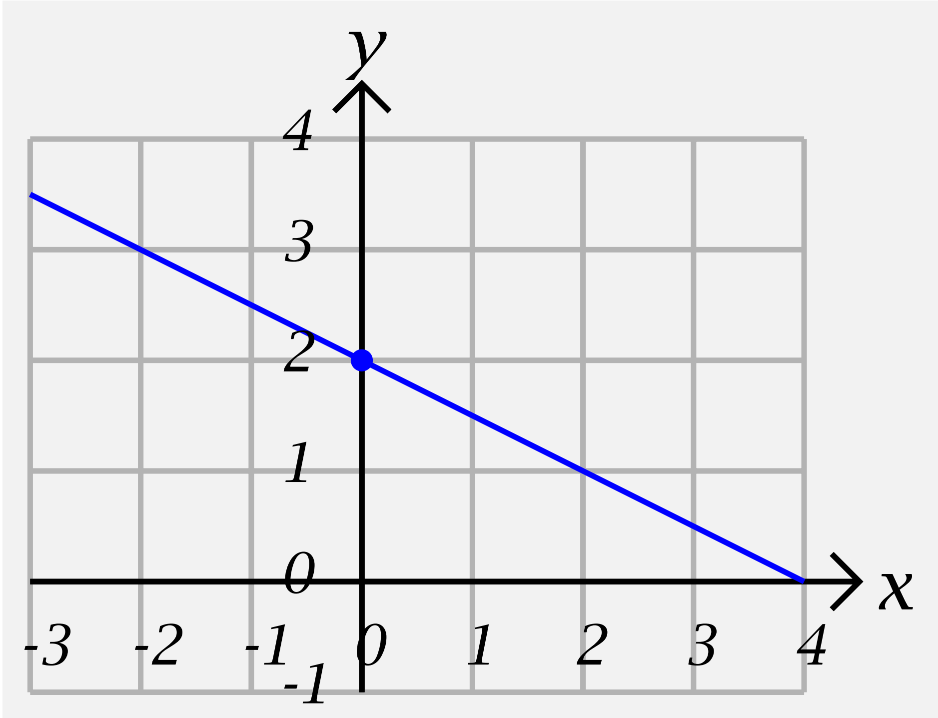
The equation of the line passing through the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This just comes from putting the two previous formulas together.

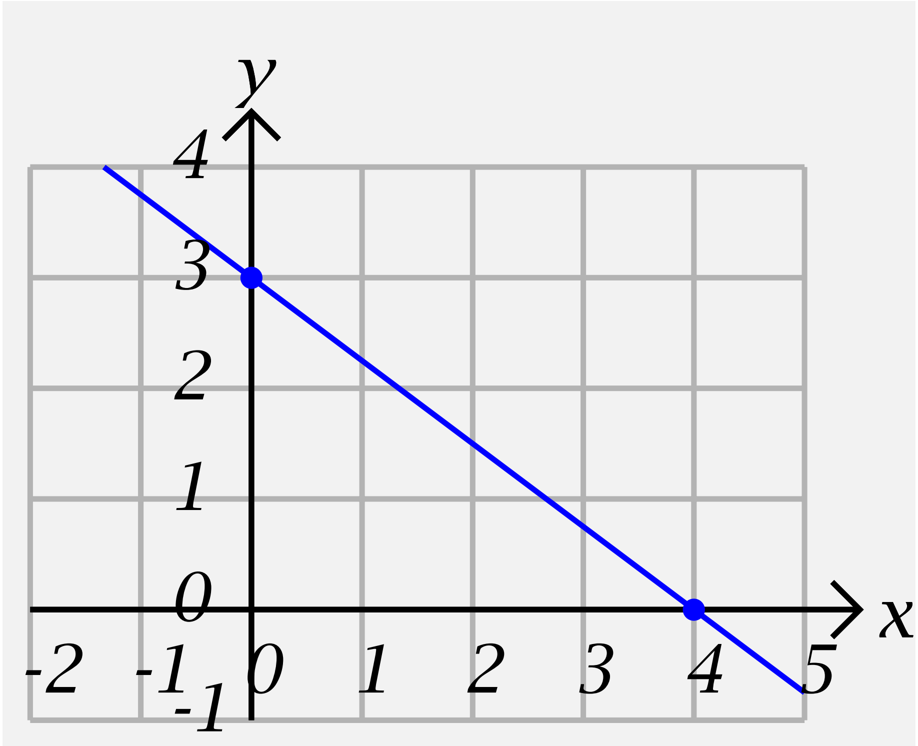
Slope-Intercept Form

The equation of the line passing through the y-axis at the point $(0, b)$ with slope m is $y = mx + b$. For example, if $m = -\frac{1}{2}$ and $b = 2$, the equation is $y = -\frac{1}{2}x + 2$



Intercept-Intercept Form

The equation of the line passing through the intercepts $(a, 0)$ and $(0, b)$ is
$\frac{x}{a} + \frac{y}{b} = 1$. For example, the equation of the line through $(4, 0)$ and $(0, 3)$ is $\frac{x}{4} + \frac{y}{3} = 1$.



General Form

Every line has infinitely many equations of the form

$$Ax + By + C = 0.$$

For any fixed line, they are non-zero multiples of each other.

Parallel & Perpendicular Lines

Two lines with slopes m_1 and m_2 are

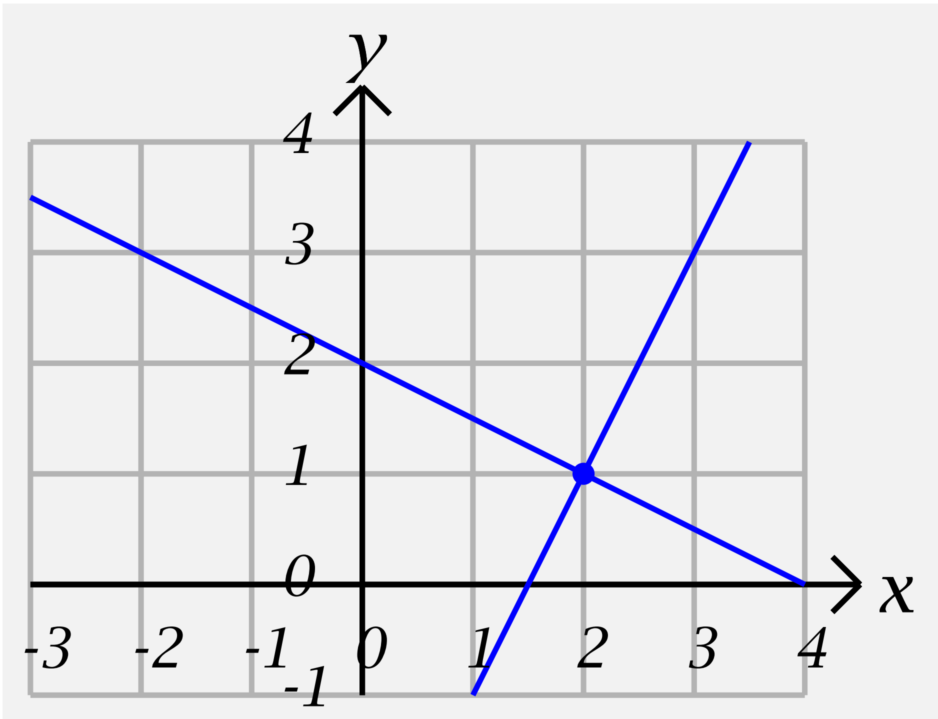
parallel if $m_1 = m_2$.

perpendicular if $m_1 m_2 = -1$.

Example: Find the equation of the line through the point $(2, 1)$ which is perpendicular to the line $y = -\frac{1}{2}x + 2$.

Solution: The slope of the perpendicular line is $-\frac{1}{-\frac{1}{2}} = 2$, so the equation of the perpendicular line is, using the Point-Slope Form:

$y - 1 = 2(x - 2)$



Distance from a Point to a Line

The distance from the point $P_0 = (x_0, y_0)$ to a line ℓ with equation
$Ax + By + C = 0$ is

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Example: Find the distance from the point $(3, 4)$ to the line with equation $y = -\frac{1}{2}x + 2$.

Solution: We must rewrite the equation of the line in General form:
$y = -\frac{1}{2}x + 2$. becomes $2y = -x + 2$ or $x + 2y - 2 = 0$, so we apply the Distance Formula with $x_0 = 3$, $y_0 = 4$, $A = 1$, $B = 2$, and $C = -2$:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{|(1)(3) + (2)(4) + (-2)|}{\sqrt{(1)^2 + (2)^2}} =$$

$$\frac{|3 + 8 - 2|}{\sqrt{1 + 4}} = \frac{9}{\sqrt{5}}$$