

Optimization

Definition: Let c be a number in the domain of a function f .

We say that $f(c)$ is a **relative maximum** of f if

$$f(c) \geq f(x) \text{ for all numbers } x \text{ in an interval } (a, b) \text{ containing } c,$$

and we say that

$f(c)$ is a **relative minimum** of f if

$$f(c) \leq f(x) \text{ for all numbers } x \text{ in an interval } (a, b) \text{ containing } c.$$

In either case we say that $f(c)$ is a **relative extremum**.

Fermat's Theorem: If f has a relative extremum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: First we recall that we know that f is continuous at c , so $\lim_{x \rightarrow c} f(x) = f(c)$.

Suppose $f(c)$ is a relative minimum. Then for all x in some interval (a, b) containing c we have $f(x) \geq f(c)$.

$$\text{If } x > c \text{ we have } \frac{f(x) - f(c)}{x - c} \geq 0, \text{ so } \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

$$\text{If } x < c \text{ we have } \frac{f(x) - f(c)}{x - c} \leq 0, \text{ so } \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0.$$

But since the two sided limit $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists (because we are given that $f'(c)$ exists), the left and right handed limits must exist and have a common value, which can be neither positive nor negative. Therefore it is 0!

Note: This means that relative extrema occur at numbers c where the first derivative $f'(c)$ is either undefined or zero. Such numbers are called **critical numbers** of f . The function values $f(c)$, if defined, are called **critical values** of f . The points $(c, f(c))$ are called **critical points** of f .

First Derivative Test:

If c is a critical number of f then:

if $f'(x)$ changes from $+$ to $-$ as x moves from the left to the right of c , then $f'(c)$ is a relative maximum;

if $f'(x)$ changes from $-$ to $+$ as x moves from the left to the right of c , then $f'(c)$ is a relative minimum;

if $f'(x)$ does not change sign at c , then $f'(c)$ is not a relative extremum.

Example 1: Let $f(x) = x^2 - 4x + 3$.

Then $f'(x) = 2x - 4 = 2(x - 2) = 0$ if $x = 2$, so 2 is a critical number.

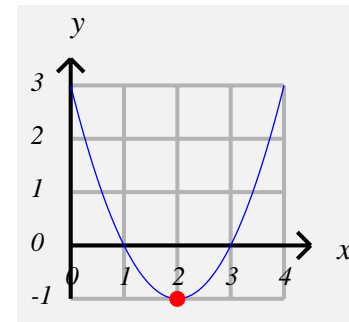
Since $f'(x) = 2(x - 2) < 0$ if $x < 2$ and

$f'(x) = 2(x - 2) > 0$ if $x > 2$,

$f'(x)$ changes from $-$ to $+$ as x increases from less than 2 to greater than 2.

Therefore $f(2) = 2^2 - 4(2) + 3 = -1$ is a **relative minimum**.

There are no other possible extrema.



Definition: Let c be a number in the domain of a function f .

We say that $f(c)$ is an **absolute maximum** of f if $f(c) \geq f(x)$ for all numbers x in the domain of f

and we say that

$f(c)$ is an **absolute minimum** of f if $f(c) \leq f(x)$ for all numbers x in in the domain of f

In either case we say that $f(c)$ is a **absolute extremum** .

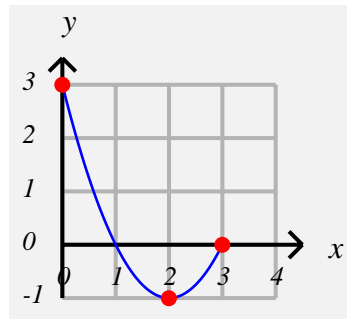
In our previous example, we conclude that the relative minimum is also the absolute minimum.

The Extreme Value Theorem: If f is continuous on the closed interval, then f has both an absolute maximum and an absolute minimum on the interval.

Example 2: In our previous example, we had $f(x) = x^2 - 4x + 3$, with the domain being $(-\infty, \infty)$, and $f'(x) = 2x - 4 = 2(x - 2) = 0$ if $x = 2$. There was an absolute minimum of -1 at $x = 2$ and no absolute maximum.

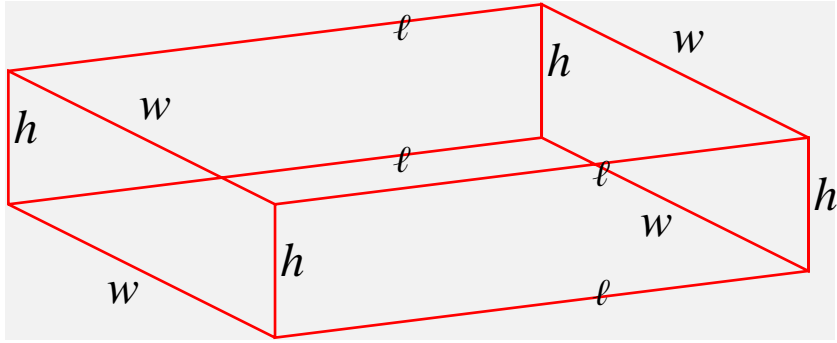
If we restrict the domain to the interval $[0, 3]$ then the absolute minimum is the same, but the the Extreme Value Theorem tells us that there must be an absolute maximum. This can only occur at the endpoints 0 and 3 of the interval, since there are no other critiacl numbers.

We have $f(0) = 3$ and $f(3) = 3^3 - 4(3) + 3 = 0$, so the absolute maximum is 3 at $x = 0$.



Example 3: A rectangular storage container with an open top is to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs $\$10$ per m^2 . Material for the sides costs $\$6$ per m^2 . Find the cost of materials for the cheapest such container.

Solution: **Sketch:**



Variables:

ℓ = length,
 w = width,
 h = height,
 V = volume,
 A_s = area of
sides,
 C_s = cost of
sides,
 A_b = area of
base,
 C_b = cost of
base,
 C = Total Cost.

Relations:

$\ell = 2w$,
 $V = \ell w h = 2w^2 h = 10 \text{ m}^3$, so
 $h = 5w^{-2}$
 $A_s = wh + \ell h + \ell h + wh =$
 $h(2w + 2\ell) = h(6w) =$
 $5w^{-2} 6w = 30w^{-1}$
 $C_s = 6A_s = 6(30w^{-1}) =$
 $180w^{-1}$
 $A_b = \ell w = 2w^2$,
 $C_b = 10(A_b) = 20w^2$,
 $C(w) = C_s + C_b =$
 $180w^{-1} + 20w^2$,
which is to be minimized.

Solution of Minimization Problem:

Differentiating with respect to w :

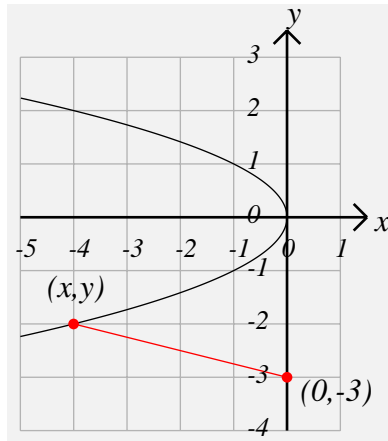
$$C'(w) = -180w^{-2} + 40w = \frac{-180}{w^2} + \frac{40w^3}{w^2} = \frac{40w^3 - 180}{w^2} = 0 \text{ if } w = \sqrt[3]{\frac{9}{2}}.$$

By the first derivative test, this gives the minimum value,

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 180\left(\sqrt[3]{\frac{9}{2}}\right)^{-1} + 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 = \frac{180}{\sqrt[3]{\frac{9}{2}}} + 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 \frac{\sqrt[3]{\frac{9}{2}}}{\sqrt[3]{\frac{9}{2}}} =$$

$$\frac{180}{\sqrt[3]{\frac{9}{2}}} + 20\frac{9}{2}\frac{1}{\sqrt[3]{\frac{9}{2}}} = \frac{270}{\sqrt[3]{\frac{9}{2}}} \doteq 163.54$$

Example 4: Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.



Solution: **Variables:** x , y , and d , the distance to a point $(x, y) = (-y^2, y)$ on the curve $x = -y^2$.

Relations: $d = \sqrt{(x - 0)^2 + (y - (-3))^2} = \sqrt{x^2 + (y + 3)^2}$

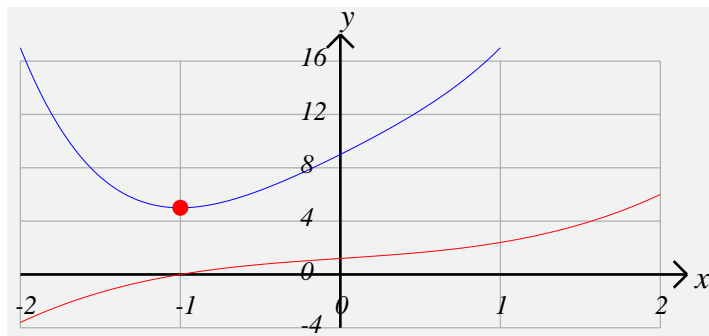
$$= \sqrt{(-y^2)^2 + (y + 3)^2} = \sqrt{y^4 + y^2 + 6y + 9}$$

We find the minimum of its square, $d^2 = g(y) = y^4 + y^2 + 6y + 9$:

We have

$$g'(y) = 4y^3 + 2y + 6 = (y + 1)(4y^2 - 4y + 6) = 0 \text{ iff } y = -1.$$

Since $4y^2 - 4y + 6 = 4(y^2 - y + \frac{3}{2}) = 4[(y - \frac{1}{2})^2 + \frac{5}{4}] > 0$, we have $g(y) < 0$ if $y < -1$ and $g(y) > 0$ if $y > -1$, so by the First Derivative test we have a relative minimum. Thus the required point is $(-1, -1)$.



The graph of g is blue, and that of g' is red.