Optimization

Definition: Let c be a number in the domain of a function f.

We say that $f(c)$ is a relative maximum of f if

$f(c) \geq f(x)$ for all numbers x in an interval (a, b) containing c,

and we say that
$f(c)$ is a **relative minimum** of f if

$f(c) \leq f(x)$ for all numbers x in an interval (a, b) containing c.
In either case we say that $f(c)$ is a **relative extremum**.

Fermat's Theorem: If f has a relative extremum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: First we recall that we know that f is continuous at c, so $\lim_{x \to c} f(x) = f(c)$.

Suppose $f(c)$ is a relative minimum. Then for all x in some interval (a, b) containing c we have $f(x) \geq f(c)$.

If $x > c$ we have $\dfrac{f(x) - f(c)}{x - c} \geq 0$, so $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$.

If $x < c$ we have $\dfrac{f(x) - f(c)}{x - c} \leq 0$, so $\displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$.

But since the two sided limit $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists (because we are given that $f'(c)$ exists), the left and right handed limits must exist and have a common value, which can be neither positive nor negative. Therefore it is 0!

Note: This means that relative extrema occur at numbers c where the first derivative $f'(c)$ is either undefined or zero. Such numbers are called **critical numbers** of f. The function values $f(c)$, if defined, are called **critical values.** of f. The points $(c, f(c))$ are called **critical points** of f.

First Derivative Test:

If c is a critical number of f then:

if $f'(x)$ changes from + to − as x moves from the left to the right of c, then $f'(c)$ is a relative maximum;

if $f'(x)$ changes from − to + as x moves from the left to the right of c, vthen $f'(c)$ is a relative minimum;

if $f'(x)$ does not change sign at c, then $f'(c)$ is not a relative extremum.

Example 1: Let $f(x) = x^2 - 4x + 3$.

Then $f'(x) = 2x - 4 = 2(x-2) = 0$ if $x = 2$, so 2 is a critical number.

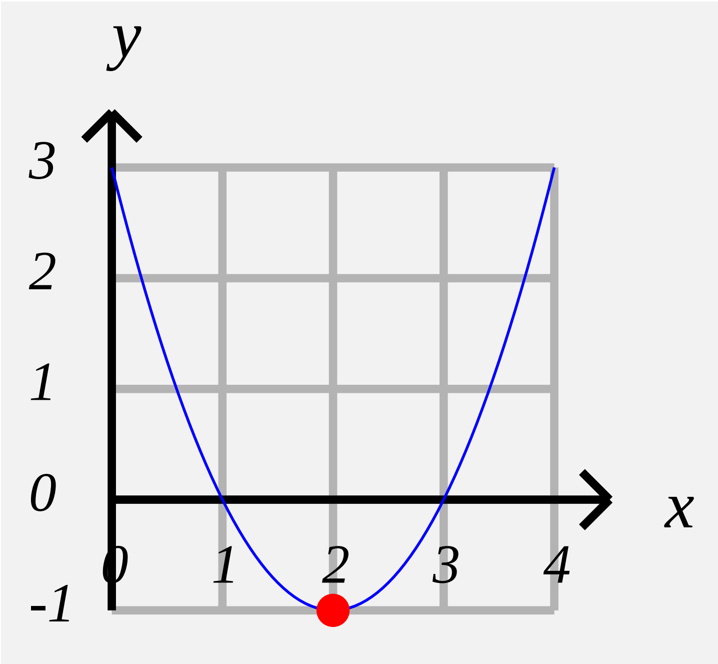
Since $f'(x) = 2(x-2) < 0$ if $x < 2$ and

$f'(x) = 2(x-2) > 0$ if $x > 2$,

$f'(x)$ changes from $-$ to $+$ as x increases from less than 2 to greater than 2.

Therefore $f(2) = 2^2 - 4(2) + 3 = -1$ is a **relative minimum**.

There are no other possible extrema.



Definition: Let c be a number in the domain of a function f.

We say that $f(c)$ is an **absolute maximum** of f if $f(c) \geq f(x)$ for all numbers x in the domain of f

and we say that

$f(c)$ is an **absolute minimum** of f if $f(c) \leq f(x)$ for all numbers x in in the domain of f

In either case we say that $f(c)$ is a **absolute extremum**.

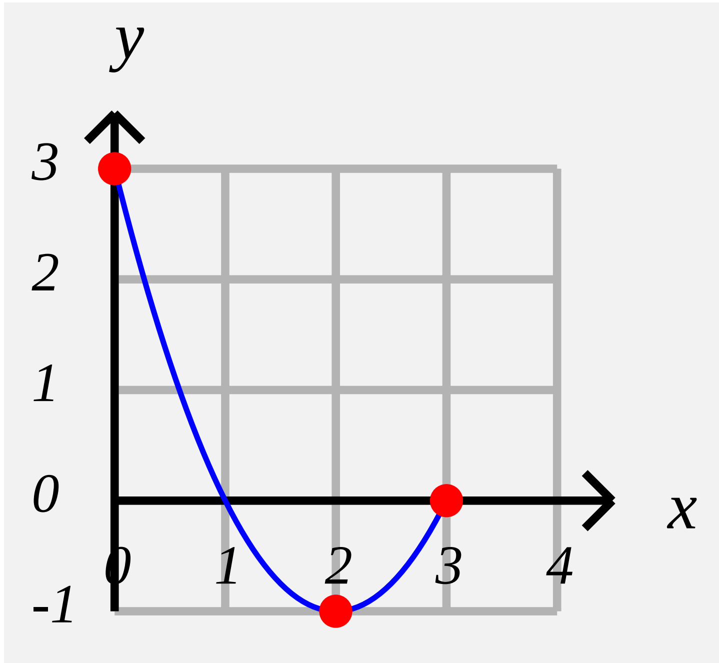
In our previous example, we conclude that the relative minimum is also the absolute minimum.

The Extreme Value Theorem: If f is continuous on the closed interval, then f has both an absolute maximum and an absolute minimum on the interval.

Example 2: In our previous example, we had $f(x) = x^2 - 4x + 3$, with the domain being $(-\infty, \infty)$, and $f'(x) = 2x - 4 = 2(x - 2) = 0$ if $x = 2$. There was an absolute minimum of -1 at $x = 2$ and no absolute maximum.

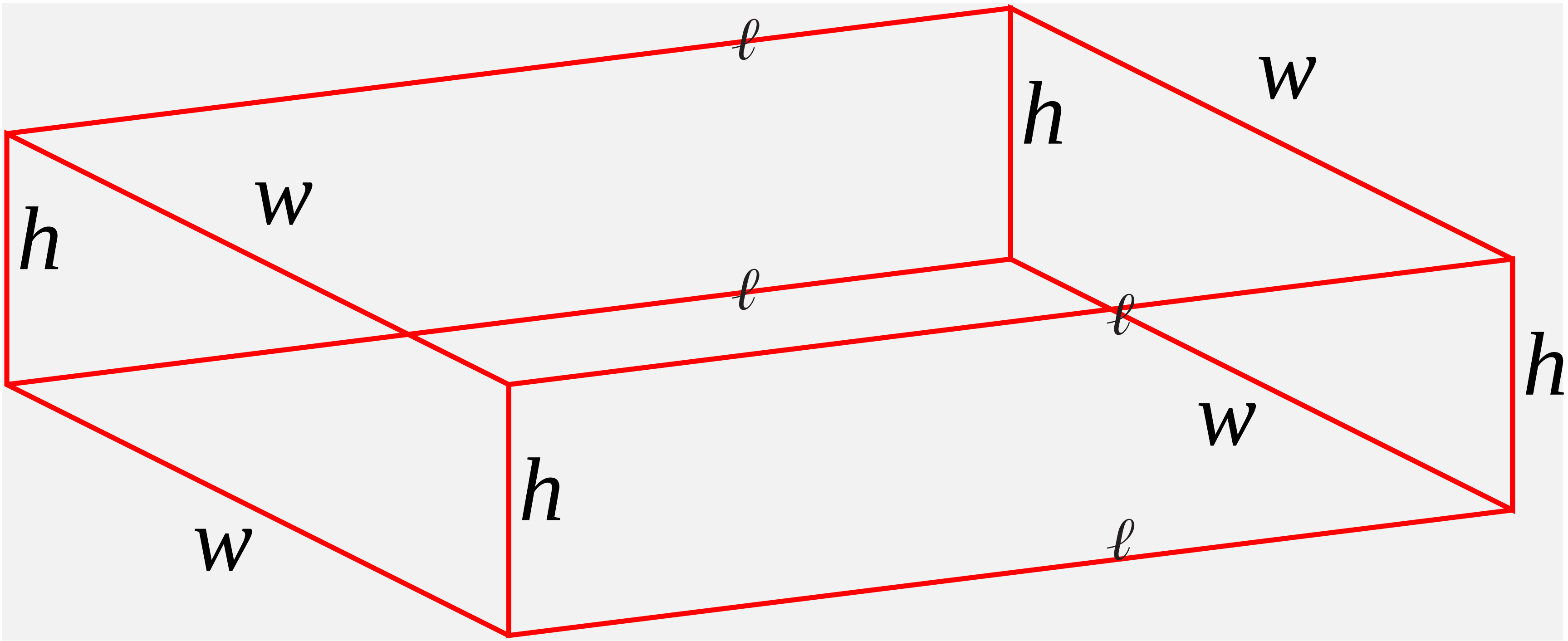
If we restrict the domain to the interval $[0, 3]$ then the absolute minimum is the same, but the the Extreme Value Theorem tells us that there must be an absolute maximum. This can only occur at the endpoints 0 and 3 of the interval, since there are no other critiacl numbers.

We have $f(0) = 3$ and $f(3) = 3^3 - 4(3) + 3 = 0$, so the absolute maximum is 3 at $x = 0$.



Problem 4.7:12. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per m^2. Material for the sides costs \$6 per m^2. Find the cost of materials for the cheapest such container.

Solution: **Sketch:**



Variables:

ℓ = length,

w = width,

h = height,

V = volume,

A_s = area of sides,

C_s = cost of sides,

A_b = area of base,

C_b = cost of base,

C = Total Cost.

Relations:

$\ell = 2w$,

$V = \ell w h = 2w^2 h = 10m^3$,so

$h = 5w^{-2}$

$A_s = wh + \ell h + \ell h + wh = h(2w + 2\ell) = h(6w) = 5w^{-2}6w = 30w^{-1}$

$C_s = 6A_s = 6(30w^{-1}) = 180w^{-1}$

$A_b = \ell w = 2w^2$,

$C_b = 10(A_b) = 20w^2$,

$C(w) = C_s + C_b = 180w^{-1} + 20w^2$,

which is to be minimized.

Solution of Minimization Problem:

Differentiating with respect to w:

$$C'(w) = -180w^{-2} + 40w = \frac{-180}{w^2} + \frac{40w^3}{w^2} = \frac{40w^3 - 180}{w^2} = 0 \text{ if}$$

$$w = \sqrt[3]{\frac{9}{2}}.$$

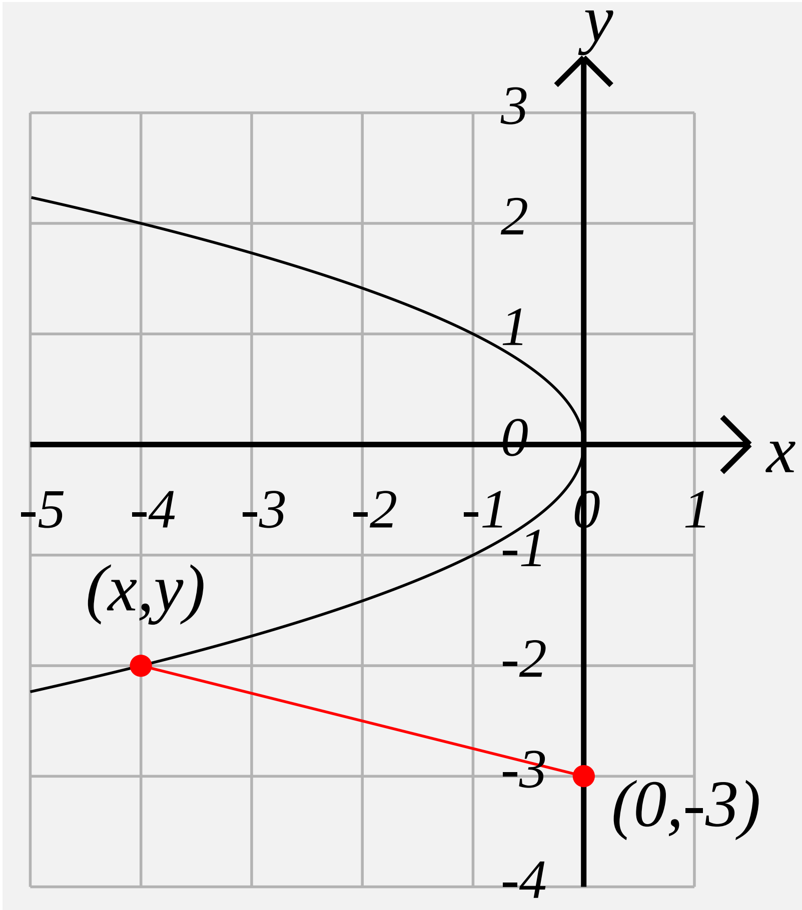
By the first derivative test, this gives the minimum value,

$$C\left(\sqrt[3]{\frac{9}{2}}\right) = 180\left(\sqrt[3]{\frac{9}{2}}\right)^{-1} + 20\left(\sqrt[3]{\frac{9}{2}}\right)^{2} =$$

$$\frac{180}{\sqrt[3]{\frac{9}{2}}} + 20\left(\sqrt[3]{\frac{9}{2}}\right)^{2}\frac{\sqrt[3]{\frac{9}{2}}}{\sqrt[3]{\frac{9}{2}}} =$$

$$\frac{180}{\sqrt[3]{\frac{9}{2}}} + 20\frac{9}{2}\frac{1}{\sqrt[3]{\frac{9}{2}}} = \frac{270}{\sqrt[3]{\frac{9}{2}}} \doteq 163.54$$

Problem 4.7:18. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.



Solution: **Variables:** x, y, and d, the distance to a point

$(x, y) = (-y^2, y)$ on the curve $x = -y^2$.

Relations: $d = \sqrt{(x-0)^2 + (y-(-3))^2} = \sqrt{x^2 + (y+3)^2}$

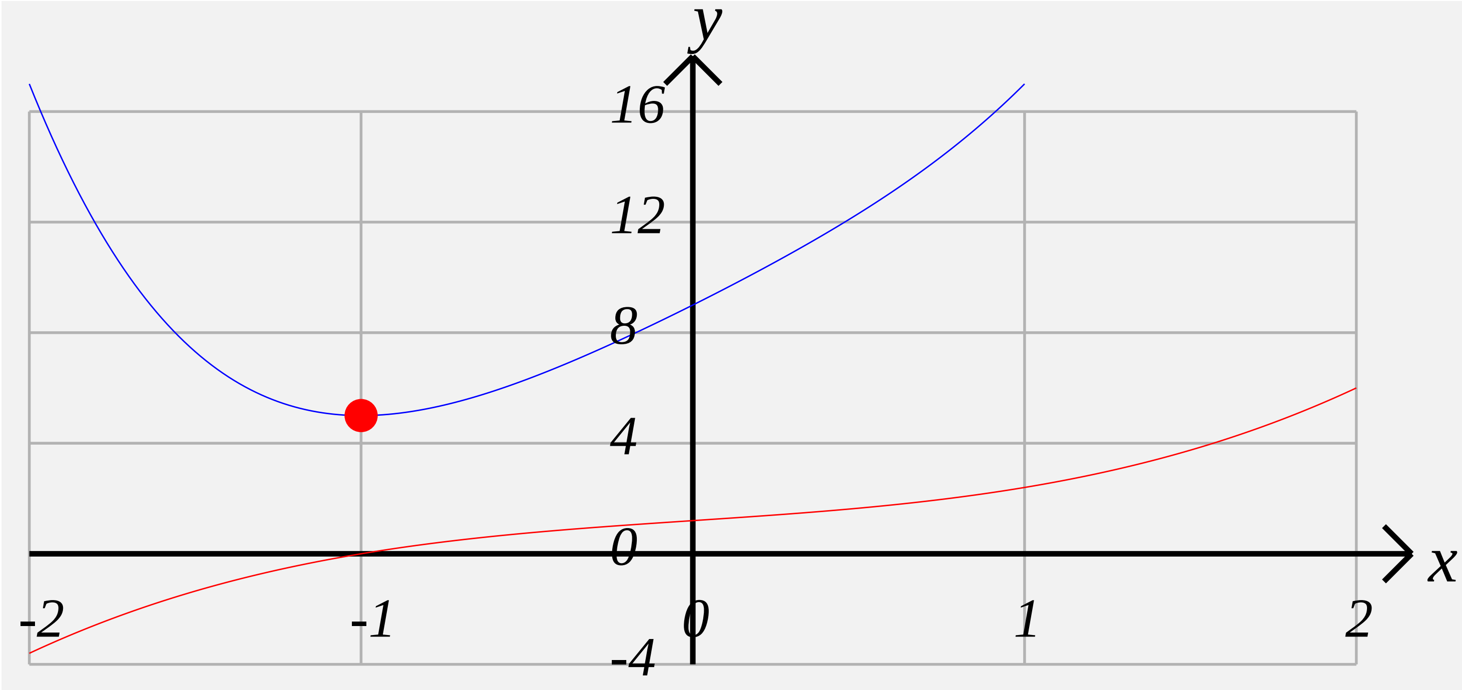
$= \sqrt{(-y^2)^2 + (y+3)^2} = \sqrt{y^4 + y^2 + 6y + 9}$

We find the minimum of its square,
$d^2 = g(y) = y^4 + y^2 + 6y + 9$:

We have

$g'(y) = 4y^3 + 2y + 6 = (y+1)(4y^2 - 4y + 6) = 0$ iff $y = -1$.

Since $4y^2 - 4y + 6 = 4(y^2 - y + \frac{3}{2}) = 4[(y - \frac{1}{2})^2 + \frac{5}{4}] > 0$, we have $g(y) < 0$ if $y < -1$ and $g(y) > 0$ if $y > -1$, so by the First Derivative test we have a relative minimum. Thus the required point is $(-1, -1)$.



The graph of g is blue, and that of g' is red.