Examples of Log Calculations

A Fundamental Principle: Since the logarithm functions are one-to-one,

if $x = y$, we must have $\ln x = \ln y$, and

if $\ln x = \ln y$ we must have $x = y$.

If a is a positive number not equal to 1, then:

if $x = y$, we will have $\log_a x = \dfrac{\ln x}{\ln a} = \dfrac{\ln y}{\ln a} = \log_a y$, and

if $\log_a x = \log_a y$ we will have

$\log_a x = \dfrac{\ln x}{\ln a} = \log_a y = \dfrac{\ln y}{\ln a}$, so $\dfrac{\ln x}{\ln a} = \dfrac{\ln y}{\ln a}$,

and since $\ln a \neq 0$, we have $\ln x = \ln y$, and therefore $x = y$.

Example 1: Solve $5^x = 3$

If 5^x is to equal 3, we must have $\ln 5^x = \ln 3$ or $x \ln 5 = \ln 3$, so we get

$$x = \frac{\ln 3}{\ln 5} \doteq \frac{1.0986123}{1.6094379} \doteq 0.68$$

Be very careful not to confuse $\dfrac{\ln 3}{\ln 5}$ with $\ln\left(\dfrac{3}{5}\right) \doteq -0.51$

Example 2: Solve $5^{3x-2} = 3^{7x+4}$ for x.

Solution: Take natural logarithms of both sides:

$\ln 5^{3x-2} = \ln 3^{7x+4}$

$(3x - 2)\ln 5 = (7x + 4)\ln 3$

$3x\ln 5 - 2\ln 5 = 7x\ln 3 + 4\ln 3$

$(3\ln 5)x - (7\ln 3)x = 4\ln 3 + 2\ln 5$

$(3\ln 5 - 7\ln 3)x = 4\ln 3 + 2\ln 5$

$$x = \frac{4\ln 3 + 2\ln 5}{3\ln 5 - 7\ln 3} \doteq -2.7$$

Example 3: Solve $3e^{x^4} = 1200$ for x.

Solution: First we simplify a little:

$e^{x^4} = 400$

and then take natural logarithms of both sides:

$\ln e^{x^4} = \ln 400$

$x^4 = \ln 400$

$$x = \pm\sqrt[4]{\ln 400} \doteq 1.56$$

Example 4: Solve $\ln(x+5) - \ln(x-4) = \ln(x+2)$ for x.

Solution: First rewrite the left hand side:

$$\ln\left(\frac{x+5}{x-4}\right) = \ln(x+2)$$

which can only be true if $\frac{x+5}{x-4} = x+2$

or $x+5 = (x+2)(x-4)$

or $x+5 = x^2 - 2x - 8$

or $x^2 - 3x - 13 = 0$

which can be solved using the Quadratic Formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-13)}}{2(1)} = \frac{3 \pm \sqrt{9+52}}{2} = \frac{3 \pm \sqrt{61}}{2} \doteq -2.4, 5.4$$

Since the given equation will have undefined terms for the negative value of x, the only possible solution is

$$x = \frac{3+\sqrt{61}}{2} \doteq 5.4$$

Example 5: Suppose the amount $R(t)$ of a radioactive isotope is given by $R(t) = 500e^{\frac{-t}{20}}$ where t is measured in hours and $R(t)$ is measured in kg. If the half-life is defined to be the time it takes the isotope to decay to $\frac{1}{2}$ of its original amount, find the half-life.

Solution: We need to solve the equation $250 = 500e^{\frac{-t}{20}}$ for t:
Taking natural logarithms, we have:

$$\ln(250) = \ln\left(500e^{\frac{-t}{20}}\right) = \ln(500) + \ln\left(e^{\frac{-t}{20}}\right) = \ln(500) + \frac{-t}{20}$$

so we must have $\ln(250) - \ln(500) = \dfrac{-t}{20}$ or

$$t = -20(\ln 250 - \ln(500)) = -20\ln\left(\frac{250}{500}\right) = -20\ln\left(\frac{1}{2}\right) = -20\,(-\ln 2) =$$

$20\ln 2 \doteq 13.86$(hours).

Example 6: Solve $\log_2 x - \log_2(x-1) = 1$

Solution: $\log_2 x - \log_2(x-1) = \log_2 \dfrac{x}{x-1} = 1$, so

$$2^{\log_2 \frac{x}{x-1}} = 2^1 = 2$$

$$\frac{x}{x-1} = 2$$

$$x = 2(x-1)$$

$$x = 2x - 2$$

$-x = -2$, $\quad x = 2$