

Examples of Log Calculations

A Fundamental Principle: Since the logarithm functions are one-to-one,

if $x = y$, we must have $\ln x = \ln y$, and

if $\ln x = \ln y$ we must have $x = y$.

If a is a positive number not equal to 1, then:

if $x = y$, we will have $\log_a x = \frac{\ln x}{\ln a} = \frac{\ln y}{\ln a} = \log_a y$, and

if $\log_a x = \log_a y$ we will have

$\log_a x = \frac{\ln x}{\ln a} = \log_a y = \frac{\ln y}{\ln a}$, so $\frac{\ln x}{\ln a} = \frac{\ln y}{\ln a}$,

and since $\ln a \neq 0$, we have $\ln x = \ln y$, and therefore $x = y$.

Example 1: Solve $5^x = 3$

If 5^x is to equal 3, we must have $\ln 5^x = \ln 3$ or $x \ln 5 = \ln 3$, so we get

$$x = \frac{\ln 3}{\ln 5} \doteq \frac{1.0986123}{1.6094379} \doteq 0.68$$

Be very careful not to confuse $\frac{\ln 3}{\ln 5}$ with $\ln\left(\frac{3}{5}\right) \doteq -0.51$

Example 2: Solve $5^{3x-2} = 3^{7x+4}$ for x .

Solution: Take natural logarithms of both sides:

$$\ln 5^{3x-2} = \ln 3^{7x+4}$$

$$(3x - 2) \ln 5 = (7x + 4) \ln 3$$

$$3x \ln 5 - 2 \ln 5 = 7x \ln 3 + 4 \ln 3$$

$$(3 \ln 5)x - (7 \ln 3)x = 4 \ln 3 + 2 \ln 5$$

$$(3 \ln 5 - 7 \ln 3)x = 4 \ln 3 + 2 \ln 5$$

$$x = \frac{4 \ln 3 + 2 \ln 5}{3 \ln 5 - 7 \ln 3} \doteq -2.7$$

Example 3: Solve $3e^{x^4} = 1200$ for x .

Solution: First we simplify a little:

$$e^{x^4} = 400$$

and then take natural logarithms of both sides:

$$\ln e^{x^4} = \ln 400$$

$$x^4 = \ln 400$$

$$x = \pm \sqrt[4]{\ln 400} \doteq 1.56$$

Example 4: Solve $\ln(x + 5) - \ln(x - 4) = \ln(x + 2)$ for x .

Solution: First rewrite the left hand side:

$$\ln\left(\frac{x + 5}{x - 4}\right) = \ln(x + 2)$$

which can only be true if $\frac{x + 5}{x - 4} = x + 2$

$$\text{or } x + 5 = (x + 2)(x - 4)$$

$$\text{or } x + 5 = x^2 - 2x - 8$$

$$\text{or } x^2 - 3x - 13 = 0$$

which can be solved using the Quadratic Formula:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-13)}}{2(1)} = \frac{3 \pm \sqrt{9 + 52}}{2} = \frac{3 \pm \sqrt{61}}{2} \doteq -2.4, 5.4$$

Since the given equation will have undefined terms for the negative value of x , the only possible solution is

$$x = \frac{3 + \sqrt{61}}{2} \doteq 5.4$$

Example 5: Suppose the amount $R(t)$ of a radioactive isotope is given by $R(t) = 500e^{-\frac{t}{20}}$ where t is measured in hours and $R(t)$ is measured in kg. If the **half-life** is defined to be the time it takes the isotope to decay to $\frac{1}{2}$ of its original amount, find the half-life.

Solution: We need to solve the equation $250 = 500e^{-\frac{t}{20}}$ for t :
Taking natural logarithms, we have:

$$\ln(250) = \ln\left(500e^{-\frac{t}{20}}\right) = \ln(500) + \ln\left(e^{-\frac{t}{20}}\right) = \ln(500) + \frac{-t}{20}$$

$$\text{so we must have } \ln(250) - \ln(500) = \frac{-t}{20} \text{ or}$$

$$t = -20(\ln 250 - \ln(500)) = -20 \ln\left(\frac{250}{500}\right) = -20 \ln\left(\frac{1}{2}\right) = -20(-\ln 2) = 20 \ln 2 \doteq 13.86(\text{hours}).$$

Example 6: Solve $\log_2 x - \log_2(x - 1) = 1$

Solution: $\log_2 x - \log_2(x - 1) = \log_2 \frac{x}{x - 1} = 1$, so

$$2^{\log_2 \frac{x}{x-1}} = 2^1 = 2$$

$$\frac{x}{x - 1} = 2$$

$$x = 2(x - 1)$$

$$x = 2x - 2$$

$$-x = -2, \quad x = 2$$