# Applications of Logarithmic Functions

## Constant Relative Rate of Change

If a non-negative variable $y$ is known to have constant relative rate of change, that is, $\dfrac{y'}{y} = k$, where $k$ is a constant, then we have a nice formula for the variable:

$y(t) = y(0)e^{kt}$, where $y(0)$ is the value of $y$ at time 0.

We can use log calculations to get some very useful information about the variable.

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## The Basic Technique: Finding Parameters from Data Points

If we are given that a variable $y$ has constant relative rate of change and two values of the variable at two different times, we can find the parameters $y(0)$ and $k$. Suppose we are given that $y(t) = y_1$ at time $t_1$ and $y(t) = y_2$ at time $t_2$. This means that the graph of $y$ passes through the two points $(t_1, y_1)$ and $(t_2, y_2)$. If $y_1 = y_2$ then the graph is a horizontal line and $k = 0$. If $y_1 < y_2$ then the variable is growing exponentially, so $k > 0$. If $y_1 > y_2$ then the variable is decaying exponentially to 0, so $k < 0$.

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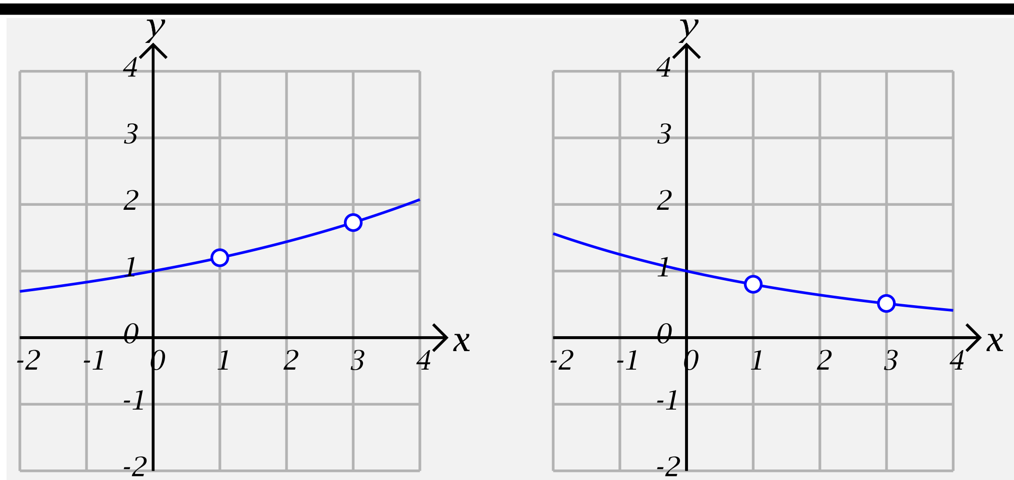

Since we know that $y(t) = y(0)e^{kt}$, we get two equations in the two unknown parameters $y(0)$ and $k$:

$y_1 = y(0)e^{kt_1}$ and $y_2 = y(0)e^{kt_2}$.

Taking ratios, we get:

$$\frac{y_2}{y_1} = \frac{y(0)e^{kt_2}}{y(0)e^{kt_1}} = \frac{e^{kt_2}}{e^{kt_1}} = e^{kt_2 - kt_1} = e^{k(t_2 - t_1)},$$

an equation involving only one unknown parameter, $k$.

Taking natural logarithms, we get:

$$\ln\left(\frac{y_2}{y_1}\right) = \ln\left(e^{k(t_2-t_1)}\right) \text{ or}$$

$$\ln y_2 - \ln y_1 = k(t_2 - t_1) \text{ or}$$

$$k = \frac{\ln y_2 - \ln y_1}{t_2 - t_1} = \frac{\ln\left(\frac{y_2}{y_1}\right)}{t_2 - t_1}.$$

We can now find $y(0)$ by substituting this value of $k$ into the equation $y_1 = y(0)e^{kt_1}$:

$$y_1 = y(0)e^{\frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1}.$$

Taking logarithms, we get:

$$\ln y_1 = \ln y(0) + \frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1, \text{ so}$$

$$\ln y(0) = \ln y_1 - \frac{\ln y_2 - \ln y_1}{t_2 - t_1} t_1 =$$

$$\frac{(t_2 - t_1)\ln y_1 - t_1 \ln y_2 + t_1 \ln y_1}{t_2 - t_1} =$$

$$\frac{t_2 \ln y_1 - t_1 \ln y_2}{t_2 - t_1} =$$

$$\frac{\ln y_1^{t_2} - \ln y_2^{t_1}}{t_2 - t_1} =$$

$$\frac{1}{t_2 - t_1} \ln \frac{y_1^{t_2}}{y_2^{t_1}}, \quad \text{so } \ln y_0 = \frac{1}{t_2 - t_1} \ln \frac{y_1^{t_2}}{y_2^{t_1}}$$

Therefore $y(0) = \left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2 - t_1}}$ and

$$y(t) = \left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2 - t_1}} e^{\frac{\ln\left(\frac{y_2}{y_1}\right)}{t_2 - t_1} t} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2 - t_1}} \left(\frac{y_2}{y_1}\right)^{\frac{t}{t_2 - t_1}} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}}\right)^{\frac{1}{t_2 - t_1}} \left(\frac{y_2^t}{y_1^t}\right)^{\frac{1}{t_2 - t_1}} =$$

$$\left(\frac{y_1^{t_2}}{y_2^{t_1}} \frac{y_2^t}{y_1^t}\right)^{\frac{1}{t_2 - t_1}} =$$

$$\left(\frac{y_2^{t - t_1}}{y_1^{t - t_2}}\right)^{\frac{1}{t_2 - t_1}}$$

Thus we have a very useful formula:

$$y(t) = \left(\frac{y_2^{t - t_1}}{y_1^{t - t_2}}\right)^{\frac{1}{t_2 - t_1}}$$

## Example 1:

A certain function $N(t)$ satisfies the exponential growth law. If $N(3) = 3000$ and $N(6) = 6000$, what is $N(4)$?

## Solution 1(Using Formula):

We have $t_1 = 3$, $N_1 = 3000$, $t_2 = 6$, and $N_2 = 6000$, so we have

$$N(t) = \left(\frac{(6000)^{t-3}}{(3000)^{t-6}}\right)^{\frac{1}{6-3}} = \left(\frac{(6000)^{t-3}}{(3000)^{t-3}(3000)^{-3}}\right)^{\frac{1}{3}} =$$

$$\left(\frac{1}{(3000)^{-3}}\right)^{\frac{1}{3}}\left(\frac{(6000)^{t-3}}{(3000)^{t-3}}\right)^{\frac{1}{3}} = \left((3000)^{-(-3)}\right)^{\frac{1}{3}}\left(\left(\frac{6000}{3000}\right)^{t-3}\right)^{\frac{1}{3}} = \left((3000)^{3}\right)^{\frac{1}{3}}(2)^{\frac{t-3}{3}} =$$

$$3000\,(2)^{\frac{t-3}{3}} = 3000\,(2)^{\frac{t}{3}-1} = 3000\,(2)^{\frac{t}{3}}\,2^{-1} = 3000\,(2)^{\frac{t}{3}}\,\frac{1}{2} = 1500\left(2^{\frac{t}{3}}\right)$$

and thus $N(4) = 1500\left(2^{\frac{4}{3}}\right) \doteq 3779.76$

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## Solution 2(Direct Computation):

We have $N(t) = N(0)e^{kt}$, and:

$$N(6) = 6000 = N(0)e^{6k}$$

$$N(3) = 3000 = N(0)e^{3k}$$

which is easily solved for $e^{3k} = 2$:

(Dividing the second equation into the first)

$$\frac{N(6)}{N(3)} = \frac{6000}{2000} = 2 = \frac{N(0)e^{6k}}{N(0)e^{3k}} = e^{3k},$$

and then for $N(0) = 1500$,
(by substituting $e^{3k} = 2$ into the second equation:)

$N(3) = 3000 = N(0)e^{3k}$

becomes $3000 = N(0)2$, so $N(0) = \dfrac{3000}{2} = 1500.$

so $N(t) = 1500e^{kt} = 1500e^{3k\frac{t}{3}} = 1500\left(e^{3k}\right)^{\frac{t}{3}} = 1500\left(2^{\frac{t}{3}}\right),$

and thus $N(4) = 1500\left(2^{\frac{4}{3}}\right) \doteq 3779.76$

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## Population Growth

Under normal conditions, populations satsify the constant relative rate of growth law.

**Example 2:** A bacteria culture starts with 500 bacteria and after 3 hours there are 8000 bacteria.

**(a)** Find an expression for the number of bacteria after $t$ hours.
**(b)** Find the number of bacteria after 4 hours.
**(c)** When will the population reach 30,000?

### Solution 1(Direct Computation):

$y(t) = y(0)e^{kt} = 500e^{kt}.$

Since $y(3) = 8000 = 500e^{k(3)}$,

we have $e^{3k} = \frac{8000}{500} = 16,$

so $3k = \ln 16$, and $k = \frac{\ln 16}{3}.$

**(a)** $y(t) = 500e^{\frac{t\ln 16}{3}} = 500\left(e^{\ln 16}\right)^{\frac{t}{3}} = 500(16)^{\frac{t}{3}}$

**(b)** $y(4) = 500(16)^{\frac{4}{3}} \doteq 20,158.75$

**(c)** $y(t) = 30000 = 500(16)^{\frac{t}{3}}$ if $\quad (16)^{\frac{t}{3}} = \dfrac{30000}{500} = 60,$

so we must have $\dfrac{t}{3}\ln 16 = \ln 60, \quad$ or $t = 3\dfrac{\ln 60}{\ln 16} \doteq 4.430168$

## Solution 2(Using Formula):

We have $t_1 = 0$, $y_1 = 500 = y(0)$, $t_2 = 3$, and $y_2 = 8000$, so we have

$$y(t) = \left(\frac{(8000)^{t-0}}{(500)^{t-3}}\right)^{\frac{1}{3-0}} = \left(\frac{(8000)^{t}}{(500)^{t}(500)^{-3}}\right)^{\frac{1}{3}} = 500\left(\frac{8000}{500}\right)^{\frac{t}{3}} = 500(16)^{\frac{t}{3}}$$

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# Radioactive Decay

Radioactive elements decay according to the formula $x(t) = x(0)e^{kt}$, where $k$ is a negative constant.

The time it takes an element to decay to half of its original value is called its half-life .

**Example 3:** For Radium-226 the value of $k$ is $-0.0004359$, assuming that the unit of time is a year. If we had 1000 grams of it (which could be very dangerous) right now, then in $t$ years we would have

$x(t) = 1000e^{-0.0004359t}$ grams left.

The table gives a few values:

| Elapsed Time | Amount Remaining | Amount Decayed |
|---|---|---|
| 0 | 1000.00 | 0.00 |
| 1 | 999.56 | 0.44 |
| 10 | 995.65 | 4.35 |
| 100 | 957.34 | 42.64 |
| 1000 | 646.66 | 353.34 |
| 1590 | 500.00 | 500.00 |

**Problem:** A radioactive isotope is weighed in a lab. At time $t_1$ there are $y_1$ grams present, and at time $t_2$ there are $y_2$ grams. Find a formula for its half-life.

## Solution (Using Formula):

$y(t) = \left(\dfrac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$, so $y(t_1) = y_1$.

We find the value of $t$ for which $y(t) = \dfrac{1}{2}y_1$:

We must have $\dfrac{1}{2}y_1 = \left(\dfrac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}}$. We take logarithms:

First, the natural logarithm of the Left Hand Side is $\ln\left(\dfrac{1}{2}y_1\right) = \ln\left(\dfrac{y_1}{2}\right) = \ln y_1 - \ln 2$,

and the natural logarithm of the Right Hand Side is

$$\ln\left(\frac{y_2^{t-t_1}}{y_1^{t-t_2}}\right)^{\frac{1}{t_2-t_1}} = \frac{1}{t_2-t_1}\ln\left(\frac{y_2^{t-t_1}}{y_1^{t-t_2}}\right) = \frac{1}{t_2-t_1}\left[\ln\left(y_2^{t-t_1}\right) - \ln\left(y_1^{t-t_2}\right)\right] =$$

$$\frac{1}{t_2-t_1}\left[(t-t_1)\ln y_2 - (t-t_2)\ln y_1\right] = \frac{1}{t_2-t_1}\left[t(\ln y_2 - \ln y_1) - t_1\ln y_2 + t_2\ln y_1\right].$$

Setting the two sides equal, we solve for $t$:

$$\ln y_1 - \ln 2 = \frac{1}{t_2-t_1}\left[t(\ln y_2 - \ln y_1) - t_1\ln y_2 + t_2\ln y_1\right]$$

$$t_2\ln y_1 - t_1\ln y_1 - (t_2-t_1)\ln 2 = t(\ln y_2 - \ln y_1) - t_1\ln y_2 + t_2\ln y_1$$

$$-t_1 \ln y_1 - (t_2 - t_1)\ln 2 = t(\ln y_2 - \ln y_1) - t_1 \ln y_2$$

$$t_1 \ln y_2 - t_1 \ln y_1 - (t_2 - t_1)\ln 2 = t(\ln y_2 - \ln y_1)$$

$$t_1(\ln y_2 - \ln y_1) - (t_2 - t_1)\ln 2 = t(\ln y_2 - \ln y_1)$$

$$t_1 - \frac{t_2 - t_1}{\ln y_2 - \ln y_1}\ln 2 = t \text{ or}$$

$$t = t_1 + \frac{t_2 - t_1}{\ln y_1 - \ln y_2}\ln 2$$

Thus the half-life is $t_{\frac{1}{2}} = \dfrac{t_2 - t_1}{\ln y_1 - \ln y_2}\ln 2$

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# Mixing of Chemicals

A tank with volume $V$ is full of water in which a chemical is dissolved at a concentration of $c_0$ grams per litre. Keeping the amount of solution in the tank constant, fresh water is added to the tank and mixed thoroughly with the solution which is drained out of the tank at the same rate, $r$ litres per minute. What is the concentration of chemical in the tank as a function of time $t$?

**Solution:** Let $y(t)$ be the amount of chemical in the tank at time $t$, and let $c(t)$ be the concentration of chemical in the tank at time $t$.

Then $c(t) = \dfrac{y(t)}{V}$,

the relative rate of change of $y$, $\left(\dfrac{y'}{y}\right)$ is the constant $-\dfrac{r}{V}$, so we have a formula for $y$ at time $t$:

$$y = y(0)e^{-\frac{r}{V}t}$$

if we let $f = \dfrac{V}{r}$, the flush time, or the length of time it takes for one volume $V$ of the tank to pass through it, we get

$$y(t) = y(0)e^{-\frac{t}{f}}$$

Expressed in terms of the concentration, we have:

$$c(t) = \frac{y(0)}{V}e^{-\frac{t}{f}} = c(0)e^{-\frac{t}{f}}$$

**Example 4:** A tank contains 1000 litres of brine with a concentration of 0.2 kg per litre. In order to dilute the solution, pure water is run into the tank at the rate of 20 litres per minute and the resulting solution, which is stirred continuously, runs out at the same rate.
(a) How many kilograms of salt remains after 30 minutes?
(b) When will the concentration be reduced to 0.1 kilograms per litre?

## Solution:

Let $y(t)$ kg be the amount of salt in the tank at time $t$, and let $c(t)$ be the concentration of salt in the tank at time $t$. Then $y(0) = 200$. We have $r = 20$ and $V = 1000$, so $y(t) = y(0)e^{-\frac{r}{V}t}$ kg $= 200e - \dfrac{20}{1000}t$ kg $= 200e^{-\frac{t}{50}}$ kg

We have $c(t) = \dfrac{y(t)\text{ kg}}{1000\ell} = \dfrac{200e^{-\frac{t}{50}}\text{ kg}}{1000\ell} = 0.2e^{-\frac{t}{50}}\dfrac{\text{kg}}{\ell}$.

(a) $y(30) = 200e^{-\frac{30}{50}} =$ $200e^{-\frac{3}{5}} \doteq 109.76$

(b) $100 = y(t) = 200e^{-\frac{t}{50}}$ if $e^{-\frac{t}{50}} = \frac{1}{2}$

or $-\dfrac{t}{50} = -\ln 2$.

Therefore $t = 50\ln 2 \doteq 34.66$

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# Newton's Law of Temperature Change

When a small object with initial temperature $T_0$ is introduced into a temperature controlled environment whose temperature, called the **ambient** temperature, is kept at $A$, **Newton's Law** says that the rate of change of of the temperature $T$ of the small object is proportional to the difference between $T$ and $A$:

$\dfrac{dT}{dt} = k(T - A)$ where $|k|$ is called the thermal coefficient of the small object.

This is equivalent to saying that $\dfrac{d(T-A)}{dt} = k(T-A)$, since $\dfrac{d(T-A)}{dt} = \dfrac{dT}{dt} - \dfrac{dA}{dt} = \dfrac{dT}{dt} - 0 = \dfrac{dT}{dt}$,
so $T - A$ has constant relative rate of change.

If we let $y(t) = T(t) - A$, we have $\dfrac{y'(t)}{y(t)} = k$, so $y(t)$ has constant relative rate of change equal to $k$, and therefore we know that $y(t) = y(0)e^{kt}$.

Since at time $t = 0$, $y(0) = T(0) - A$, we have

$y(t) = T(t) - A = (T(0) - A)e^{kt}$

or

$T(t) = A + (T_0 - A)e^{kt}$.

Since $k < 0$, $\lim_{t\to\infty} e^{kt} = 0$, and therefore

$\lim_{t\to\infty} T = A$. Because of this, we often write $T_\infty$ instead of $A$:

$T(t) = T_\infty + (T_0 - T_\infty)e^{kt}$

In practice, this equation is assumed to be known to hold and the problem is to use physical observations of temperature to determine the constants that appear in it and then to determine when a certain target temperature will be attained.

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**Example 5:** A thermometer is taken from a room where the temperature is $20^\circ$ to the outdoors, where the temperature is $5^\circ$. After one minute the thermometer reads $12^\circ$. Use Newton's Law of Cooling to answer the following questions:
(a) What will the reading on the thermometer be after one more minute?
(b) When will the thermometer read $6^\circ$?

**Solution:** We have $T(t) = T_\infty + (T_0 - T_\infty)e^{kt} = 5 + 15e^{kt}$

Since $T(1) = 12$, we have $12 = 5 + 15e^k$, so $e^k = \frac{7}{15}$.

Thus $T(t) = 5 + 15\left(\frac{7}{15}\right)^t$.

(a) $T(2) = 5 + 15\left(\frac{7}{15}\right)^2 = 5 + \frac{49}{15} =$ $\boxed{8\frac{4}{15}}$

(b) $T(t) = 5 + 15\left(\frac{7}{15}\right)^t = 6$

if $\left(\frac{7}{15}\right)^t = \frac{1}{15}$ or

$t \ln\left(\frac{7}{15}\right) = -\ln 15$ or

$$t = -\frac{\ln 15}{\ln\left(\frac{7}{15}\right)} = -\frac{\ln 15}{\ln 7 - \ln 15} = \boxed{\frac{\ln 15}{\ln 15 - \ln 7} \doteq 3.553}$$

**Example 6:** A turkey is taken from a freezer and thawed before being roasted. Using a meat thermometer, the internal temperature of the thawed turkey is found to be 3°. It is placed in an oven set at 200° and after 30 minutes its internal temperature is observed to be 20°. The turkey is deemed to be cooked when its internal temperature reaches 85°. How long will this take?

**Solution:** We have $T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so $e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

We want to know the value of $t$ for which $T(t) = 85$: so we solve the equation

$85 = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$ for $t$:

$$85 - 200 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$ and then take logs:

$$\ln \frac{115}{197} = \ln \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$\ln 115 - \ln 197 = \frac{t}{30} \ln \left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$, so:

$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090} \doteq 5.977$, so $t \doteq 5.977 \times 30$ minutes $\doteq 3$ hours.