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We can use log calculations to get some very useful information about the variable.

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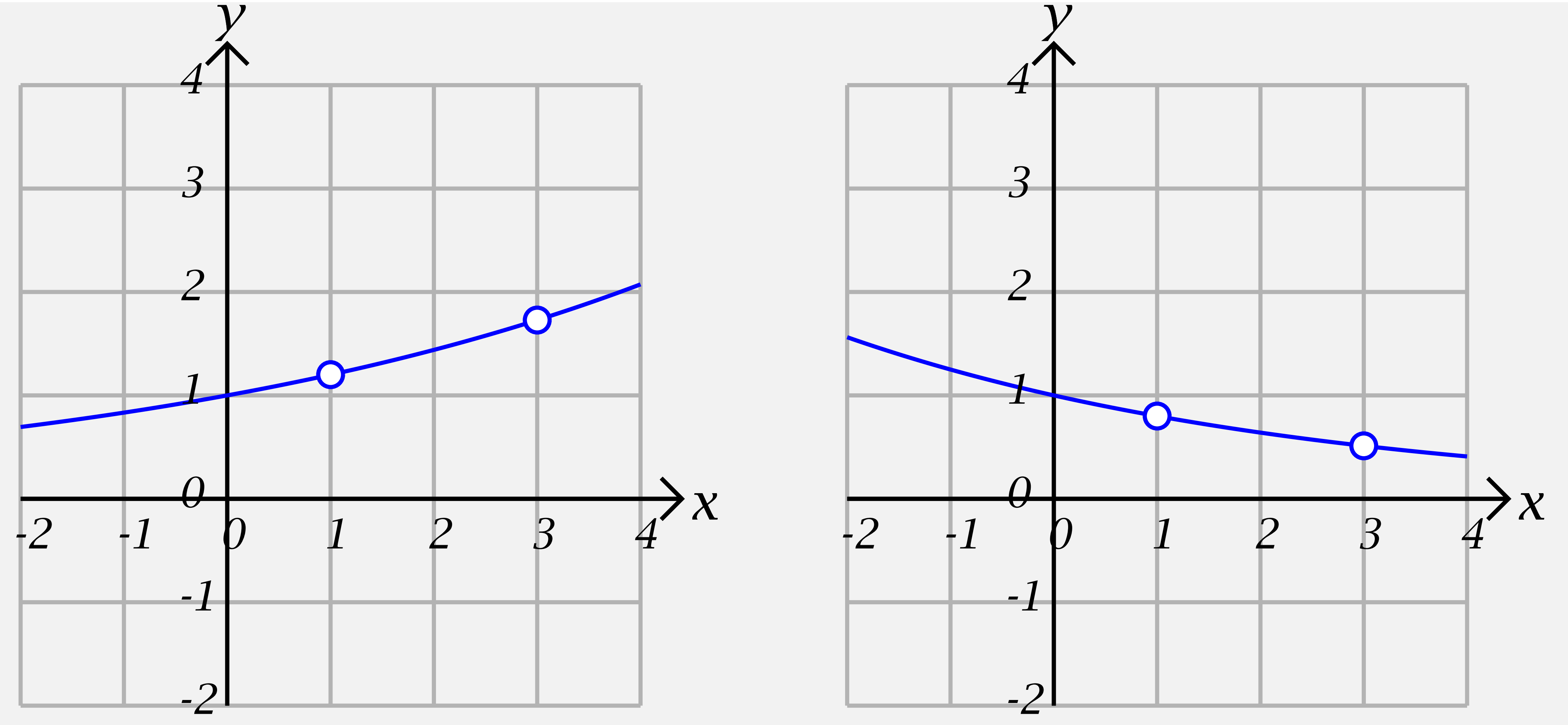
y
4
3
2
1
0
x
-2
-1
0
1
2
3
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an equation involving only one unknown parameter, k.

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and thus $N(4) = 1500\left(2^{\frac{4}{3}}\right) \doteq 3779.76$

Solution 2(Direct Computation):

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Population Growth

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(a) Find an expression for the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) When will the population reach 30,000?

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Radioactive Decay

Radioactive elements decay according to the formula $x(t) = x(0)e^{kt}$, where k is a negative constant.

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The time it takes an element to decay to half of its original value is called its half-life.

Example 3: For Radium-226 the value of k is -0.0004359, assuming that the unit of time is a year. If we had 1000 grams of it (which could be very dangerous) right now, then in t years we would have

$x(t) = 1000e^{-0.0004359t}$ grams left.

The table gives a few values:

Elapsed Time	Amount Remaining	Amount Decayed
0	1000.00	0.00
1	999.56	0.44
10	995.65	4.35
100	957.34	42.64
1000	646.66	353.34
1590	500.00	500.00

Problem: A radioactive isotope is weighed in a lab. At time t_1 there are y_1 grams present, and at time t_2 there are y_2 grams. Find a formula for its half-life.

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Setting the two sides equal, we solve for t:

$$\ln y_1 - \ln 2 = \frac{1}{t_2-t_1}\left[t(\ln y_2 - \ln y_1) - t_1\ln y_2 + t_2\ln y_1\right]$$

$$t_2\ln y_1 - t_1\ln y_1 - (t_2-t_1)\ln 2 = t(\ln y_2 - \ln y_1) - t_1\ln y_2 + t_2\ln y_1$$

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Mixing of Chemicals

A tank with volume V is full of water in which a chemical is dissolved at a concentration of c_0 grams per litre. Keeping the amount of solution in the tank constant, fresh water is added to the tank and mixed thoroughly with the solution which is drained out of the tank at the same rate, r litres per minute.

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Example 4: A tank contains 1000 litres of brine with a concentration of 0.2 kg per litre. In order to dilute the solution, pure water is run into the tank at the rate of 20 litres per minute and the resulting solution, which is stirred continuously, runs out at the same rate.

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(a) How many kilograms of salt remains after 30 minutes?

(b) When will the concentration be reduced to 0.1 kilograms per litre?

Solution:

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so $T - A$ has constant relative rate of change.

If we let $y(t) = T(t) - A$, we have $\frac{y'(t)}{y(t)} = k$, so $y(t)$ has constant relative rate of change equal to k, and therefore we know that $y(t) = y(0)e^{kt}$.

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$$T(t) = T_\infty + (T_0 - T_\infty)e^{kt}$$

In practice, this equation is assumed to be known to hold and the problem is to use physical observations of temperature to determine the constants that appear in it and then to determine when a certain target temperature will be attained.

Example 5: A thermometer is taken from a room where the temperature is 20° to the outdoors, where the temperature is 5°. After one minute the thermometer reads 12°. Use Newton's Law of Cooling to answer the following questions:

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Example 5: A thermometer is taken from a room where the temperature is 20° to the outdoors, where the temperature is 5°. After one minute the thermometer reads 12°. Use Newton's Law of Cooling to answer the following questions:

(a) What will the reading on the thermometer be after one more minute?

(b) When will the thermometer read 6°?

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(a) $T(2) = 5 + 15\left(\frac{7}{15}\right)^2 = 5 + \frac{49}{15} = 8\frac{4}{15}$

(b) $T(t) = 5 + 15\left(\frac{7}{15}\right)^t = 6$

if $\left(\frac{7}{15}\right)^t = \frac{1}{15}$ or

$$t \ln\left(\tfrac{7}{15}\right) = -\ln 15 \text{ or}$$

$$t \ln\left(\frac{7}{15}\right) = -\ln 15 \text{ or}$$

$$t = -\frac{\ln 15}{\ln\left(\frac{7}{15}\right)}$$

$$t \ln\left(\frac{7}{15}\right) = -\ln 15 \text{ or}$$

$$t = -\frac{\ln 15}{\ln\left(\frac{7}{15}\right)} = -\frac{\ln 15}{\ln 7 - \ln 15}$$

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$$t \ln\left(\tfrac{7}{15}\right) = -\ln 15 \text{ or}$$

$$t = -\frac{\ln 15}{\ln\left(\tfrac{7}{15}\right)} = -\frac{\ln 15}{\ln 7 - \ln 15} = \frac{\ln 15}{\ln 15 - \ln 7} \doteq 3.553$$

Example 6: A turkey is taken from a freezer and thawed before being roasted. Using a meat thermometer, the internal temperature of the thawed turkey is found to be 3°. It is placed in an oven set at 200° and after 30 minutes its internal temperature is observed to be 20°. The turkey is deemed to be cooked when its internal temperature reaches 85°. How long will this take?

Solution: We have

$T(t) =$

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} =$

Solution: We have

$$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} =$$

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Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$,

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so

$e^{30k} = \frac{20-200}{-197} =$

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so
$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so
$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so

$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

We want to know the value of t for which $T(t) = 85$: so we solve the equation

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200-3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so
$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

We want to know the value of t for which $T(t) = 85$: so we solve the equation

$85 = 200 - 197\left(\dfrac{180}{197}\right)^{\frac{t}{30}}$ for t:

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so
$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

We want to know the value of t for which $T(t) = 85$: so we solve the equation

$85 = 200 - 197\left(\dfrac{180}{197}\right)^{\frac{t}{30}}$ for t:

$85 - 200 = -197\left(\dfrac{180}{197}\right)^{\frac{t}{30}}$

Solution: We have

$T(t) = T_\infty - (T_\infty - T_0)e^{kt} = 200 - (200 - 3)e^{kt} = 200 - 197e^{kt} =$

Since $T(30) = 20$, we have $20 = 200 - 197e^{30k}$, so
$e^{30k} = \frac{20-200}{-197} = \frac{180}{197}$.

Thus $T(t) = 200 - 197\left(\frac{180}{197}\right)^{\frac{t}{30}}$.

We want to know the value of t for which $T(t) = 85$: so we solve the equation

$85 = 200 - 197\left(\dfrac{180}{197}\right)^{\frac{t}{30}}$ for t:

$$85 - 200 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

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$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

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$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} =$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090}$$

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197],$$ so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090} \doteq 5.977,$$ so

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090} \doteq 5.977$$, so

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

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$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090} \doteq 5.977$$, so

$t \doteq 5.977 \times 30$ minutes

$$-115 = -197\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{-115}{-197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\frac{115}{197} = \left(\frac{180}{197}\right)^{\frac{t}{30}}$$ and then take logs:

$$\ln\frac{115}{197} = \ln\left(\frac{180}{197}\right)^{\frac{t}{30}}$$

$$\ln 115 - \ln 197 = \frac{t}{30}\ln\left(\frac{180}{197}\right) = \frac{t}{30}[\ln 180 - \ln 197]$$, so:

$$\frac{t}{30} = \frac{\ln 115 - \ln 197}{\ln 180 - \ln 197} \doteq \frac{4.745 - 5.283}{5.193 - 5.283} \doteq \frac{-0.538}{-0.090} \doteq 5.977$$, so

$t \doteq 5.977 \times 30$ minutes $\doteq 3$ hours.