Derivatives of Exponentials & Logarithms

$$\frac{d}{dx}\left(e^x\right) = e^x$$ and $$\frac{d}{dx}\left(a^x\right) = (\ln a)a^x$$

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$ and $$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{(\ln a)x}$$

We may get more general formulas by using the Chain Rule:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$ and

$$\frac{d}{dx}\left(a^{f(x)}\right) = (\ln a) a^{f(x)} f'(x)$$

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$ and

$$\frac{d}{dx}(\log_a f(x)) = \frac{f'(x)}{(\ln a) f(x)} = \frac{1}{\ln a}\frac{f'(x)}{f(x)}$$

The quantity $\frac{f'(x)}{f(x)}$ is called the relative rate of change of the function f, and is very important in practical applications.

Example 1: Find the derivative of $y = e^{3x}$.

Solution: $y' = e^{3x}(3x)' = e^{3x}3 = \boxed{3e^{3x}}$

Example 2: Find the derivative of $y = e^{x^3}$.

Solution: $y' = e^{x^3}(x^3)' = e^{x^3}(3x^2) = \boxed{3x^2e^{x^3}}$

Example 3: Find the derivative of $y = 3^{x^2}$.

Solution: $y' = (\ln 3)3^{x^2}(x^2)' = (\ln 3)3^{x^2}(2x) =$

$(2x\ln 3)3^{x^2}$

Example 4: Find the derivative of $y = \ln(x^5 + x^3 + 1)$.

Solution:

$$y' = \frac{1}{x^5 + x^3 + 1}(x^5 + x^3 + 1)' = \frac{1}{x^5 + x^3 + 1}(5x^4 + 3x^2) =$$

$$\frac{x^2(5x^2 + 3)}{x^5 + x^3 + 1}$$

Logarithmic Differentiation

One of the most important uses of the natural logarithm function is in the computation of derivatives of functions which are made up of products, quotients and powers of more elementary functions. We use the three basic arithmetic properties of the logarithm to simplify the function.

Example 5: Find y' if $y = \dfrac{(x+1)^4}{(x-1)^3(x+4)^7}$

Taking logarithms of both sides of the equation, we get

$$\ln y = \ln\left(\frac{(x+1)^4}{(x-1)^3(x+4)^7}\right) \text{ or}$$

$$\ln y = 4\ln(x+1) - 3\ln(x-1) - 7\ln(x+4)$$

which we now differentiate:

$$\frac{y'}{y} = 4\frac{1}{x+1} - 3\frac{1}{x-1} - 7\frac{1}{x+4}$$

which we need only simplify slightly to get y' in a usable form:

$$y' = y\left[\frac{4}{x+1} - \frac{3}{x-1} - \frac{7}{x+4}\right]$$

Example 6: find y' if $y = x^x$

Solution: Since, by definition, $x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}$, the function is only defined for $x > 0$, and must always be positive.

We have $\ln y = \ln(x^x) = x \ln x$, so

$$\frac{y'}{y} = (x \ln x)' = (x)'(\ln x) + x(\ln x)' = (1)\ln x + x\frac{1}{x} = 1 + \ln x,$$

so $y' = x^x(1 + \ln x).$

Negative x

There will be occasions when we wish to apply logarithms and deal with negative values of the variables concerned.

Of course, $\ln x$ is undefined if $x \leq 0$. However, $\ln|x|$ *is* defined if $x < 0$.

Let us then find the derivative of $\ln|x|$ for non-zero x.

If $x > 0$, it is of course $\dfrac{1}{x}$.

If $x < 0$, then $|x| = -x$, so $\ln|x| = \ln(-x)$, and we can apply the Chain Rule:

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x},$$

so we have the important formulas

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ if } x \neq 0$$

and

$$\frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)} \text{ if } f(x) \neq 0$$