Inverse Functions

One-to-One Functions

A function is "one-to-one" if it never takes on the same value twice; i.e.,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

One-to-one functions are functions which do not achieve any value more than once on a specified interval. Any function which is strictly increasing or strictly decreasing on an interval is one-to-one on that interval.

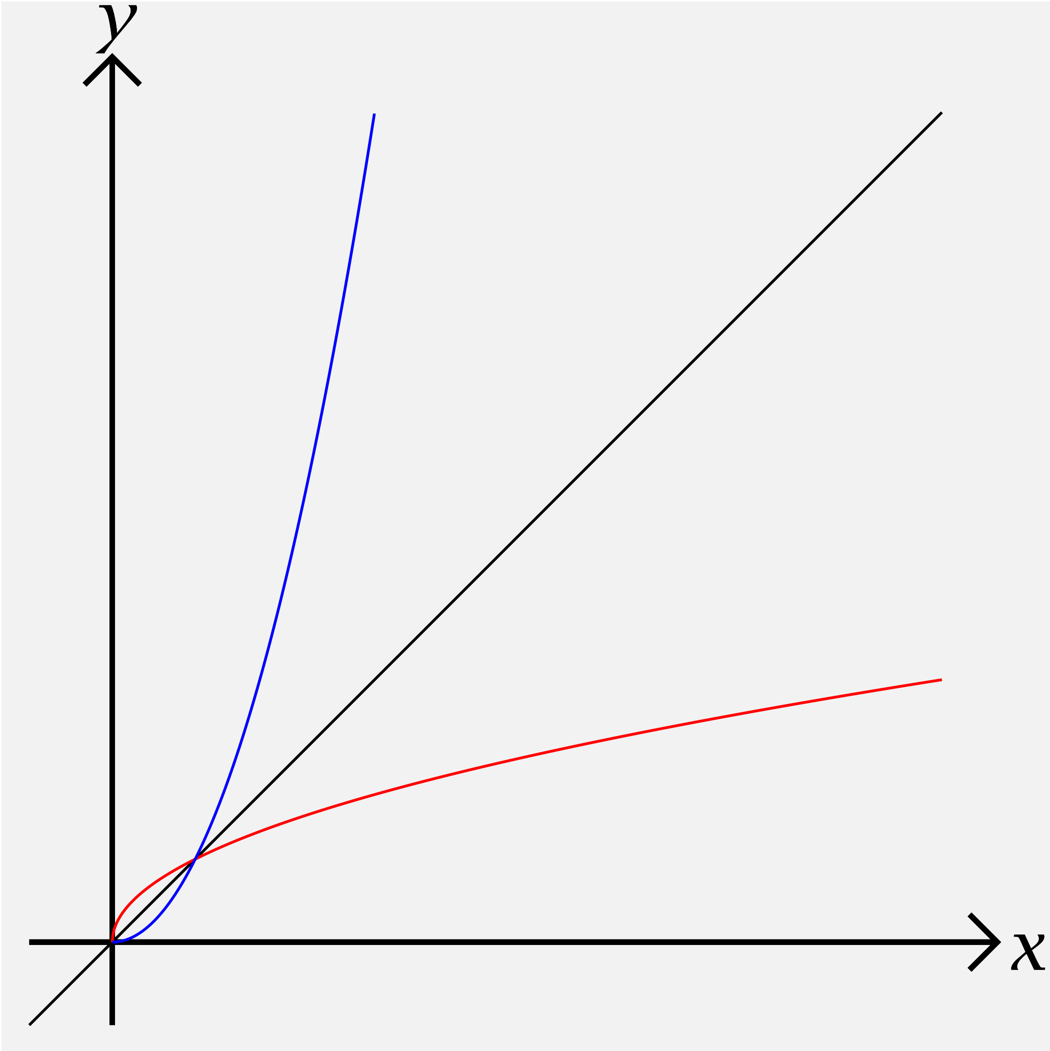
Recall the Vertical Line Test which tells us whether or not a curve can be the graph of a function:

If no vertical line intersects the curve more than once, then the curve is the graph of a function.

There is also the **Horizontal Line Test**

If no horizontal line intersects the curve more than once, then the curve is the graph of a one-to-one function.

Since the reflection of the graph of a one-to-one function f in the line $y = x$ clearly passes the Vertical Line Test, it is the graph of a new function, which we call the inverse of f and which we denote by f^{-1} or by f^{inv}. The domain of f^{-1} is the range of f and vice-versa.



It is best to look at a Java applet .

We always have the so-called

Cancellation Equations:

$f^{-1}(f(x)) = x$ for all x in the domain of f,

and $f(f^{-1}(y)) = y$ for all y in the range of f.

In most cases, it is impossible to explicitly calculate a formula for the inverse function. We will look at a number of extremely important cases where this can be done.

Example 1: $f(x) = mx + b$, with $m \neq 0$ is 1:1, so we take the equation $y = mx + b$ and solve for x:

$$x = \frac{y - b}{m}.$$

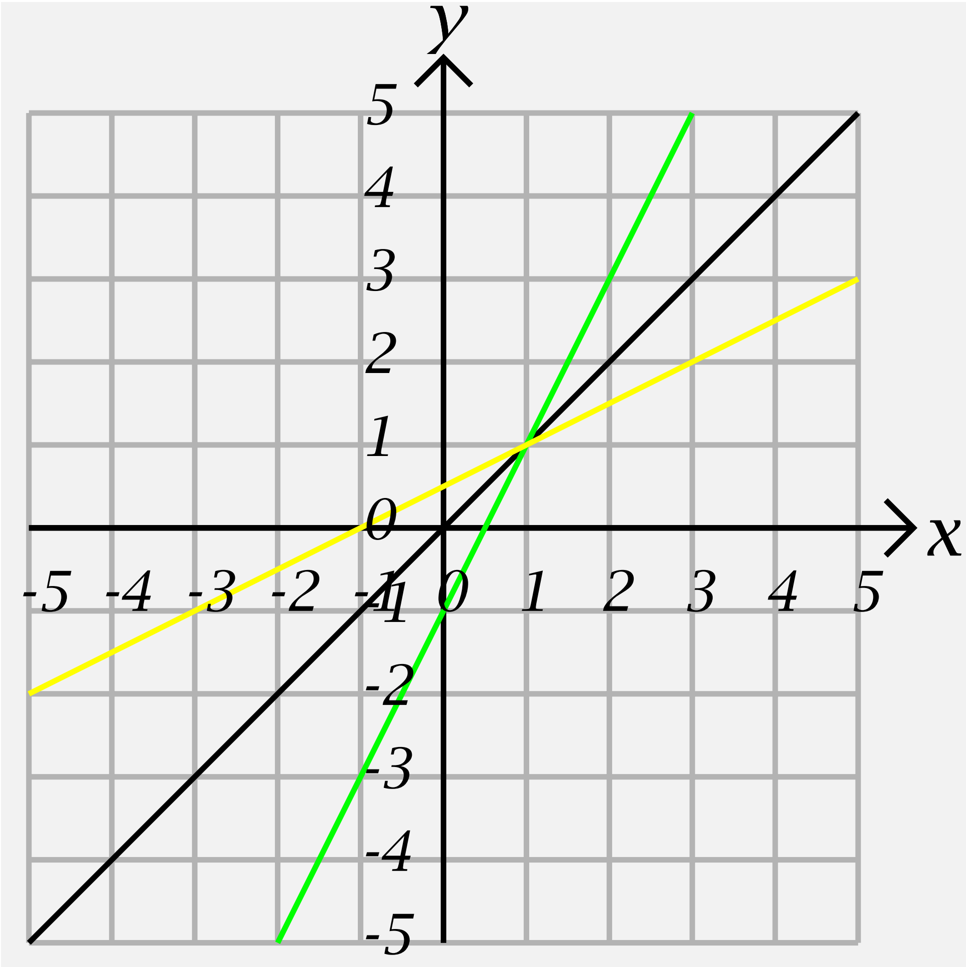
Interchanging x and y in this equation, we get $y = \dfrac{x - b}{m}$. Thus the formula for

f^{-1} is $f^{-1}(x) = f^{inv}(x) = \dfrac{x - b}{m}$. We then check to see that the cancellation laws are satisfied:

$$f\left(f^{-1}(x)\right) = m\left(f^{-1}(x)\right) + b = m\left(\frac{x-b}{m}\right) + b = x - b + b = x$$

Specific Example: Letting $m = 2$ and $b = -1$ we have

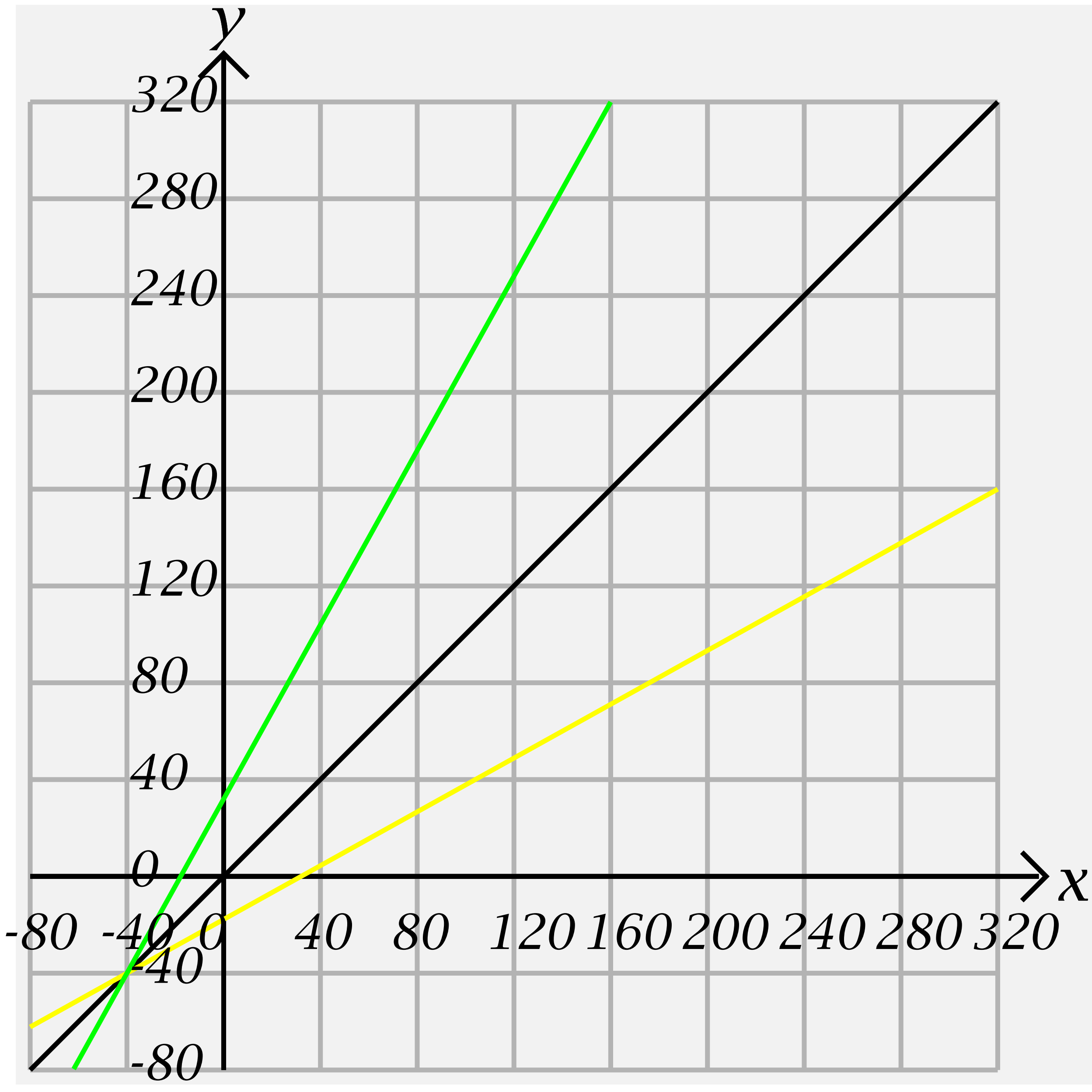
$$f(x) = 2x - 1 \text{ and } f^{inv}(x) = \frac{x-(-1)}{2} = \frac{x+1}{2}$$



Another Specific Example: Celsius-Fahrenheit Conversion

Letting $m = \frac{5}{9}$ and $b = -\frac{160}{9}$ we have $f(x) = \frac{5}{9}x - \frac{160}{9}$ and $f^{inv}(x) = \frac{x - (-\frac{160}{9})}{\frac{5}{9}} = \frac{9x + 160}{5} = \frac{9}{5}x + 32$.

These two formulas are usually written as $C = \frac{5}{9}F - \frac{160}{9}$ and $F = \frac{9}{5}C + 32$, where F and C represent the Fahrenheit and Celsius temperatures respectively.

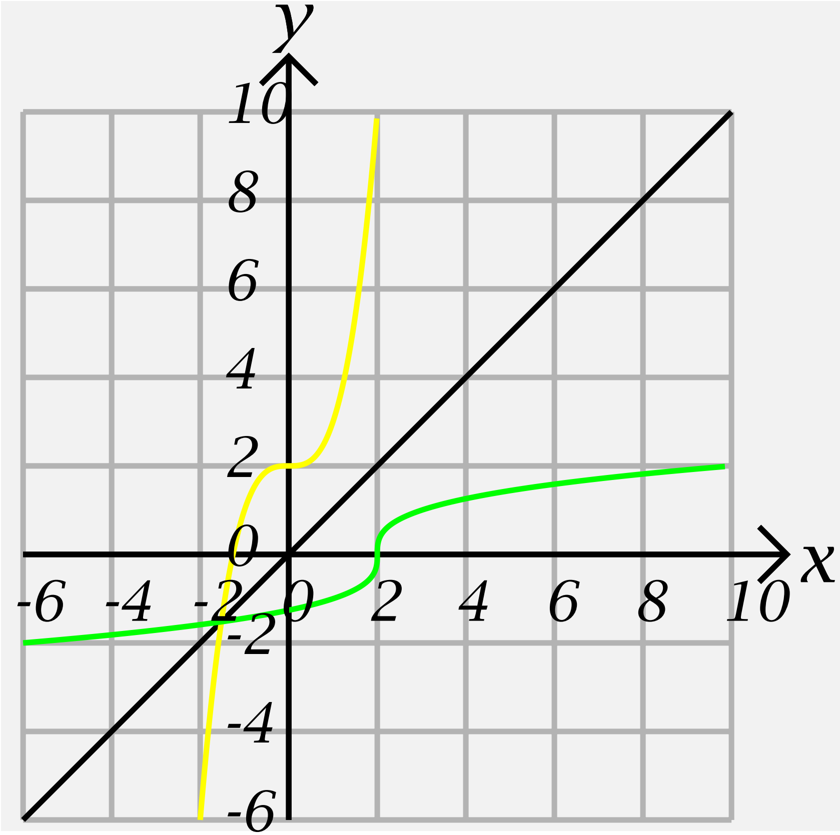

y
x
320
280
240
200
160
120
80
40
0
-40
-80
-80 -40 0 40 80 120 160 200 240 280 320

Example 2: $f(x) = x^3 + 2$ is 1:1, so we take the equation $y = x^3 + 2$ and solve for x:

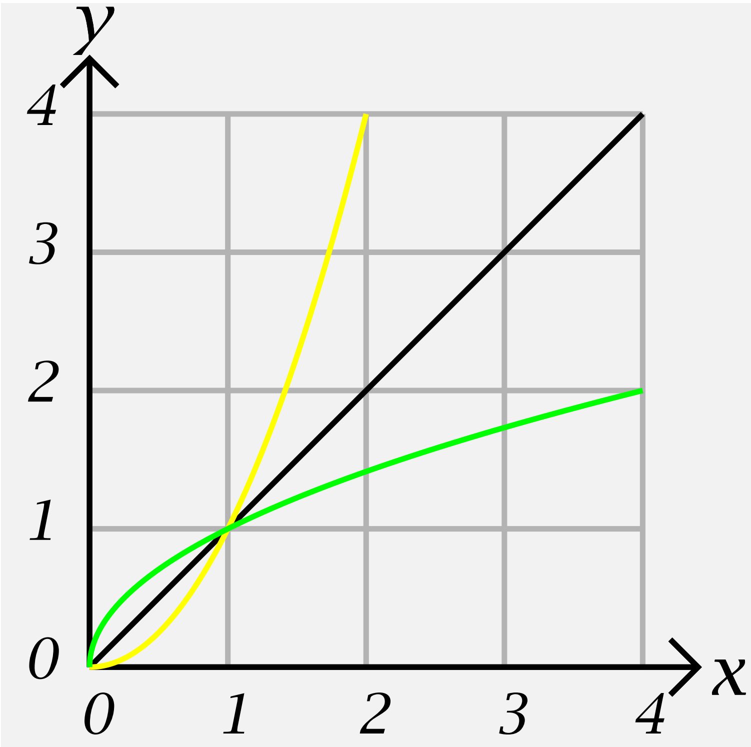
$y - 2 = x^3$, so $x = (y-2)^{\frac{1}{3}}$.

Interchanging x and y in this equation, we get $y = (x-2)^{\frac{1}{3}}$. Thus the formula for f^{-1} is $f^{-1}(x) = f^{inv}(x) = (x-2)^{\frac{1}{3}}$. We then check to see that the cancellation laws are satisfied:

$$f\left(f^{-1}(x)\right) = \left(f^{-1}(x)\right)^3 + 2 = \left((x-2)^{\frac{1}{3}}\right)^3 + 2 = (x-2) + 2 = x$$


y
x
10
8
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6
8
10

Example 3: $f(x) = x^2$ is 1:1 on the interval $[0, \infty)$, and its inverse is used to define $\sqrt{x}$.



Example 4: $f(x) = x^2$ is 1:1 on the interval $(-\infty, 0]$, and its inverse is $-\sqrt{-x}$.