

Limit Laws

We let \lim stand for any one of the limits $\lim_{x \rightarrow a}$, $\lim_{x \rightarrow a^-}$, $\lim_{x \rightarrow a^+}$, $\lim_{x \rightarrow -\infty}$, or $\lim_{x \rightarrow +\infty}$, and suppose c is a constant and that $\lim f(x)$ and $\lim g(x)$ exist. Then

$$(1) \lim [f(x) + g(x)] = \lim f(x) + \lim g(x)$$

$$(2) \lim [f(x) - g(x)] = \lim f(x) - \lim g(x)$$

$$(3) \lim [f(x)g(x)] = \lim f(x) \lim g(x)$$

$$(4) \lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \text{ if } \lim g(x) \text{ is not equal to } 0.$$

$$(5) \lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}, \text{ if } n \text{ is a positive integer and } \lim f(x) \geq 0 \text{ is also required if } n \text{ is even.}$$

$$(6) \text{ If } n \text{ is a positive integer, } \lim (f(x))^n = (\lim f(x))^n$$

$$(7) \lim c f(x) = c \lim f(x)$$

$$(8) \text{ If } c \text{ is a constant then } \lim c = c \text{ for any } a.$$

The next three limit laws do not apply to limits as $x \rightarrow \pm\infty$:

$$(9) \lim_{x \rightarrow a} x = a, \lim_{x \rightarrow a^-} x = a, \lim_{x \rightarrow a^+} x = a, \text{ for any } a.$$

$$(10) \lim_{x \rightarrow a} x^n = a^n, \lim_{x \rightarrow a^-} x^n = a^n, \lim_{x \rightarrow a^+} x^n = a^n, \text{ for any } a, \text{ if } n \text{ is a positive integer.}$$

$$(11) \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}, \lim_{x \rightarrow a^-} \sqrt[n]{x} = \sqrt[n]{a}, \lim_{x \rightarrow a^+} \sqrt[n]{x} = \sqrt[n]{a}, \text{ for any } a > 0, \text{ if } n \text{ is a positive integer.}$$

(12) If $f(x) = g(x)$ for all x except possibly $x = a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ if either limit exists.

(13) If $f(x)$ is a polynomial or a rational function and a is in the domain of f , then $\lim_{x \rightarrow a} f(x) = f(a)$

Examples:

Example 1: $\lim_{x \rightarrow -1} [x^5 - 3x^3 + 1] (x^2 - 2) = [(-1)^5 - 3(-1)^3 + 1] ((-1)^2 - 2) =$

$$[-1 - 3(-1) + 1] (1 - 2) = [-1 + 3 + 1] (-1) = (3)(-1) = -3$$

Example 2:

$$\lim_{x \rightarrow 25} \frac{\sqrt{x}}{x + 25} = \frac{\sqrt{25}}{25 + 25} = \frac{5}{50} = \frac{1}{10}$$

Example 3:

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \left(\frac{\sqrt{x} - 3}{x - 9} \right) \left(\frac{\sqrt{x} + 3}{\sqrt{x} + 3} \right) = \lim_{x \rightarrow 9} \frac{(\sqrt{x})^2 - 3^2}{(x - 9)(\sqrt{x} + 3)} =$$

$$\lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\lim_{x \rightarrow 9} (\sqrt{x} + 3)} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

Example 4:

$$\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(x + 2)} = \lim_{x \rightarrow 2} \frac{1}{x + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$

Example 5:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2$$

Example 6: $\lim_{x \rightarrow 0} |x|$

DANGER! $|x|$ is a function defined by a multicolumn formula, and different formulas apply for $|x|$ on either side of the limiting value of x , namely 0. We must calculate the two one-sided limits:

$$\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = (-0) = 0$$

$$\lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} x = 0$$

Since the left and right hand limits are equal, we can conclude that the two-sided limit does exist and equals the common value 0.

Example 7:

$$\lim_{x \rightarrow 0} \frac{|x|}{x}$$

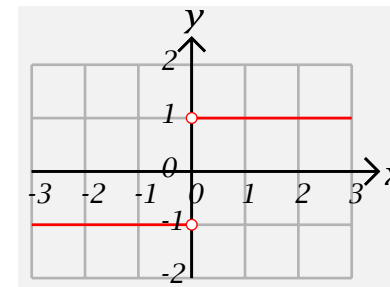
Note that the limiting value of x , 0, is not in the domain of the function $\frac{|x|}{x}$.

Again, $|x|$ is a function defined by a multicolumn formula, and different formulas apply for $|x|$ on either side of the limiting value of x , so again we must calculate the two one-sided limits:

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} -1 = -1$$

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

Since the left and right hand limits are unequal, we can conclude that the two-sided limit does not exist.



Limits of Rational Functions

If $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \rightarrow a} f(x) = f(a) = \frac{p(a)}{q(a)} \quad \text{if } q(a) \neq 0$$

If $q(a) = 0$, then there are a number of possibilities:

if $p(a) \neq 0$, we could have $\lim_{x \rightarrow a} f(x) = +\infty$, or $\lim_{x \rightarrow a} f(x) = -\infty$, or
 $\lim_{x \rightarrow a^-} f(x) = -\infty$ and $\lim_{x \rightarrow a^+} f(x) = +\infty$, or $\lim_{x \rightarrow a^-} f(x) = +\infty$ and $\lim_{x \rightarrow a^+} f(x) = -\infty$.

If $p(a) = 0$, it is possible, but not guaranteed, that $\lim_{x \rightarrow a} f(x)$ can exist.

Example 8: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$.

Here $a = 2$, $p(x) = x^2 - 4$, $q(x) = x - 2$, so $p(a) = p(2) = 0$, $q(a) = q(2) = 0$. We have to do some algebraic manipulation to compute the limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} x + 2 = 2 + 2 = 4.$$

Rational Functions at $\pm \infty$

Theorem: If $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then:

(1) if the degree of $p(x)$ is less than the degree of $q(x)$, $\lim_{x \rightarrow \pm \infty} f(x) = 0$,

(2) if the degree of $p(x)$ is greater than the degree of $q(x)$, $\lim_{x \rightarrow \pm \infty} f(x) = \pm \infty$,

(3) if the degree of $p(x)$ equals the degree of $q(x)$, $\lim_{x \rightarrow \pm \infty} f(x) = \frac{a_0}{b_0}$,

where a_0 and b_0 are the leading coefficients of $p(x)$ and $q(x)$.

Example 9: The Floor and Ceiling Functions .

We define $\lfloor x \rfloor$, the **floor of x** , to be the largest integer that is less than or equal to x , and $\lceil x \rceil$, the **ceiling of x** , to be the least integer that is greater than or equal to x . In other words, the floor function rounds x down to the nearest integer and the ceiling function rounds x up to the nearest integer. They are both examples of step functions. They both have two-sided limits at non-integer values:

If a is not an integer, then $\lim_{x \rightarrow a} \lfloor x \rfloor = \lfloor a \rfloor$, and $\lim_{x \rightarrow a} \lceil x \rceil = \lceil a \rceil$, but if a is an integer, things are different:

$$\lim_{x \rightarrow a^-} \lfloor x \rfloor = \lfloor a \rfloor - 1,$$

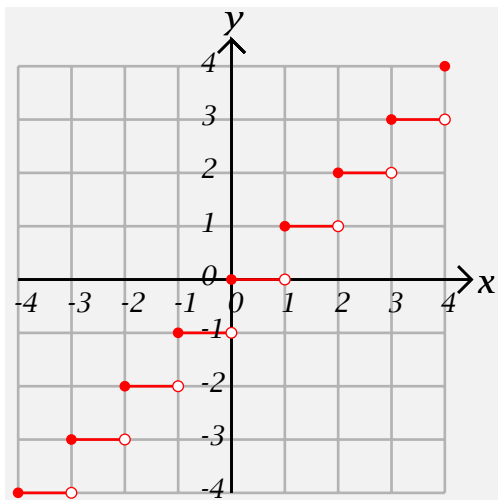
$$\lim_{x \rightarrow a^+} \lfloor x \rfloor = \lfloor a \rfloor$$

and

$$\lim_{x \rightarrow a^-} \lceil x \rceil = \lceil a \rceil,$$

$$\lim_{x \rightarrow a^+} \lceil x \rceil = \lceil a \rceil + 1$$

The Floor Function:



The Ceiling Function:

