Tangent Lines

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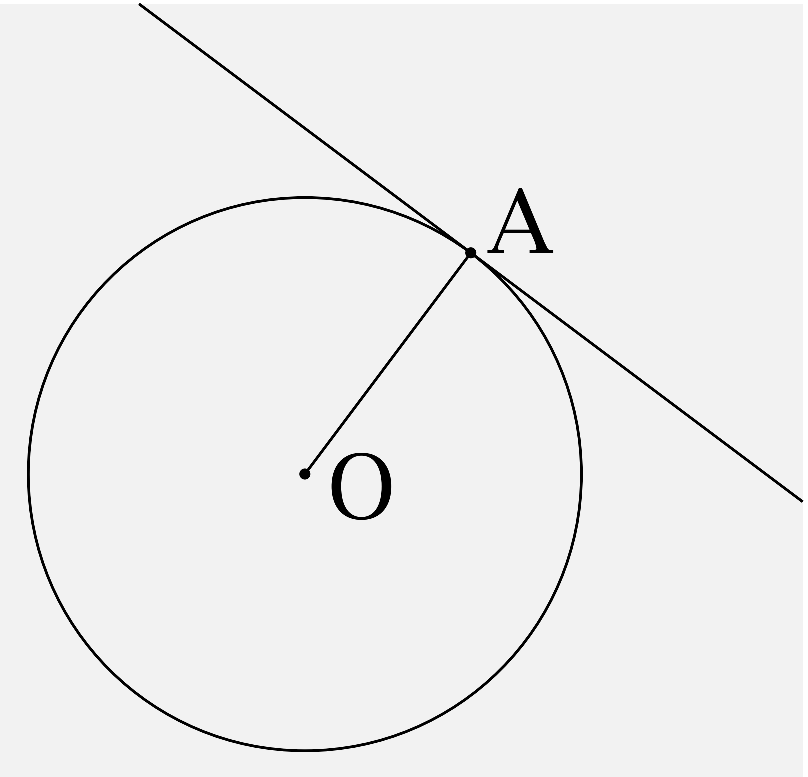

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The underlying concept needed to understand the correct definitions of tangent lines and velocities is that of the limit of a function .

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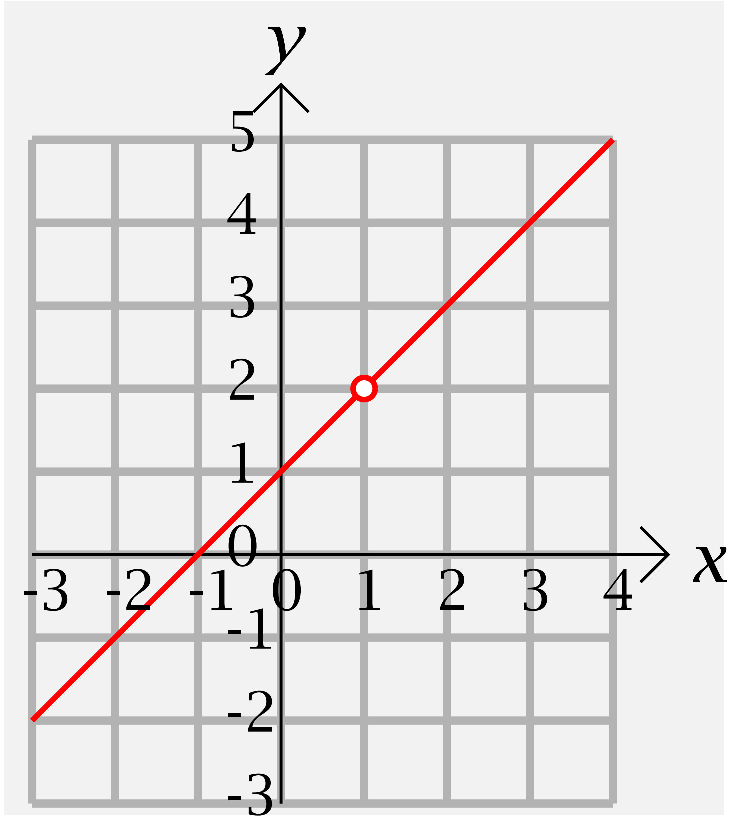

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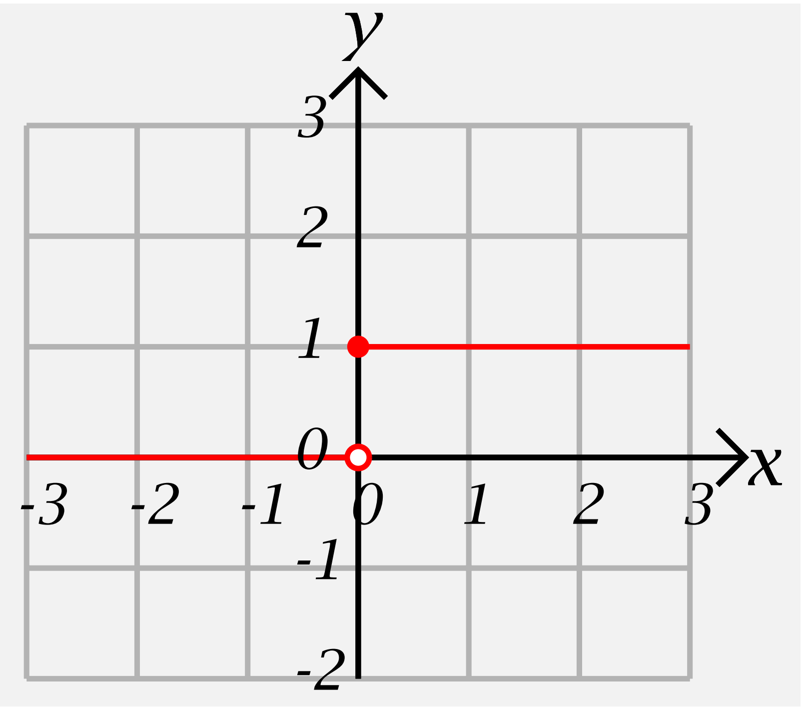

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Sometimes we will refer to the ordinary limit $\lim_{x \to a} f(x)$ as a "two-sided limit".

Fact: If $\lim_{x \to a} f(x)$ exists then both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ exist.

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so the two-sided limit exists everywhere except at $a = 0$.

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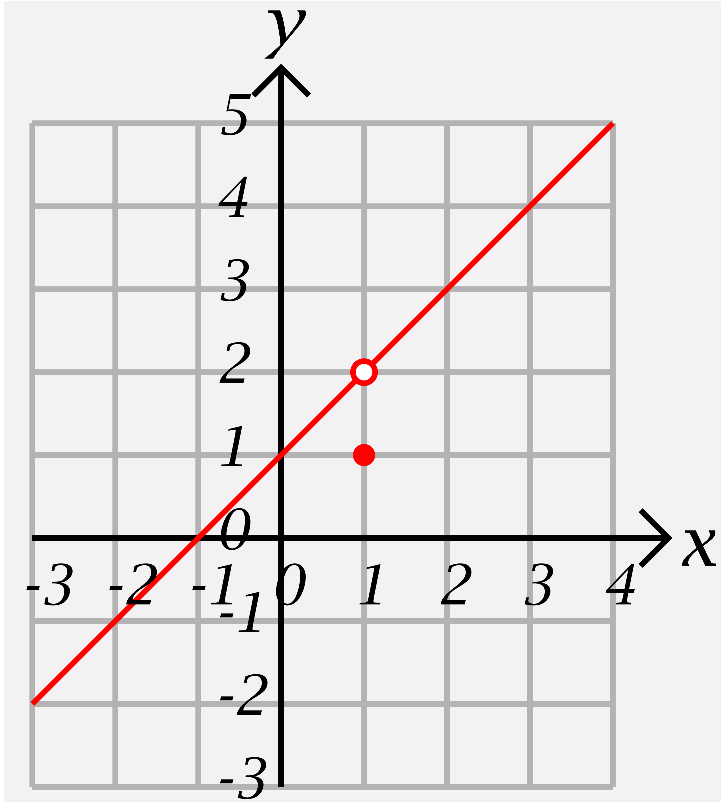

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However, we have $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to 0^+} f(x) = +\infty$, and we can also talk about the **limits at infinity**:
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There is another type of asymptote not covered in Math 101 called a **slant asymptote:** if $y = mx + b$ and $\lim_{x \to -\infty} |f(x) - (mx + b)| = 0$, then the line $y = mx + b$ is a slant asymptote.