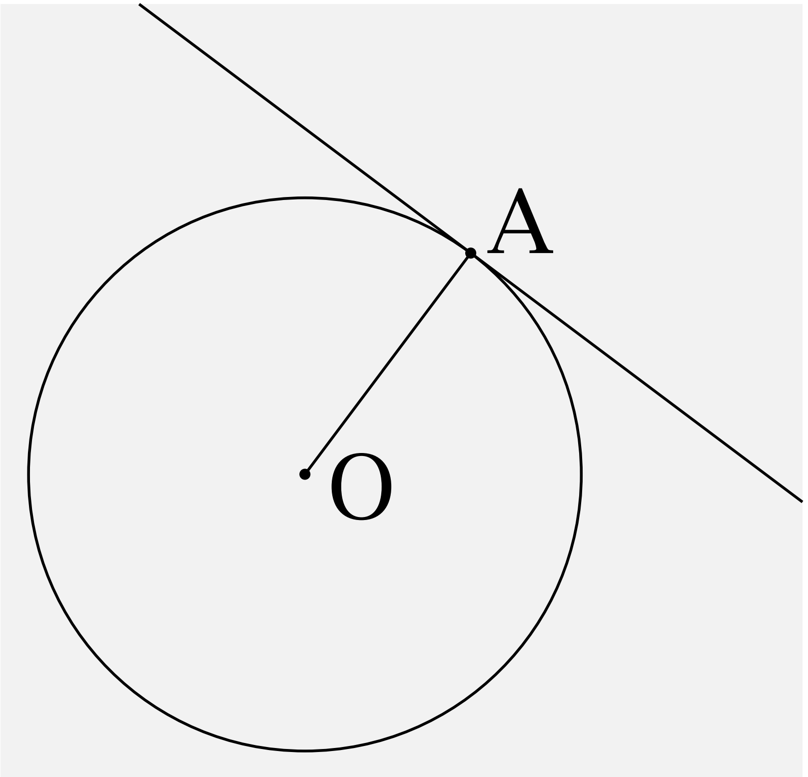
Tangent Lines

The concept of the tangent line to a circle dates back at least to the early days of Greek geometry, that is over 2500 years. The tangent line to a circle with centre **O** at a point **A** is simply constucted by taking the perpendicular to the line through **O** and **A** at **A**.



The conic sections — parabolas, ellipses, and hyperbolas, were discovered and studied about 2000 years ago. The idea of a tangent line to such a curve was simple enough, since these curves can all be viewed as being shadows of circles, but the geometric construction of these tangent lines with the tools of the day was very difficult. It was only with the discovery, or invention, of Calculus in the 17th century that the proper tools became available, and it then became possible to define the concept of a tangent lines at points on a very large class of curves.

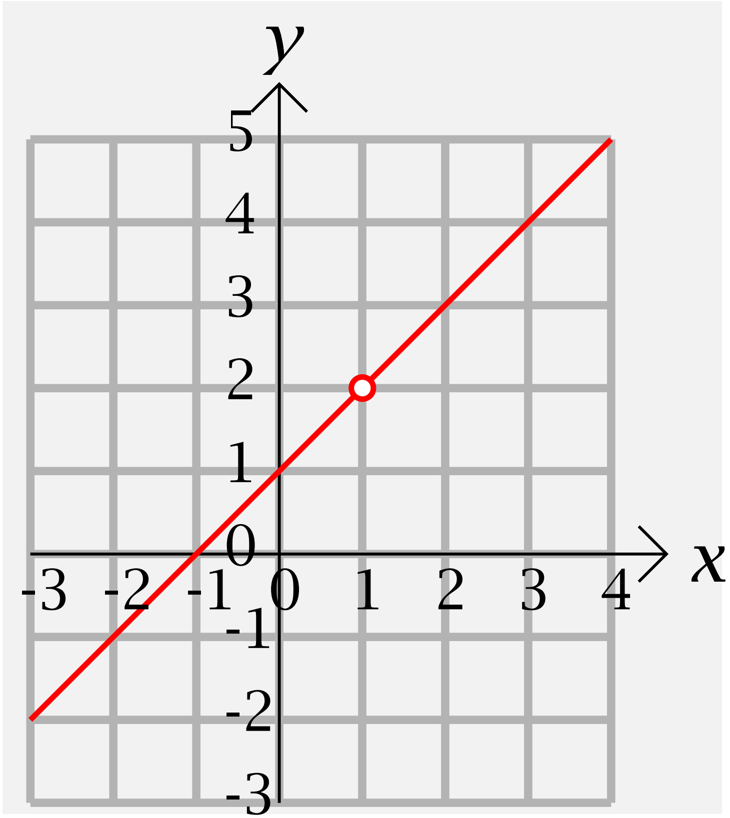
Velocity

Again, the history of the concept of velocity begins with the Greeks, especially with Zeno's four paradoxes. The questions raised were not successfully resolved until after not only Calculus as a tool had been constructed, but the careful definition of the real numbers had been completed about 1870-1880.

The underlying concept needed to understand the correct definitions of tangent lines and velocities is that of the limit of a function .

Limits

Consider the function $f(x) = \dfrac{x^2-1}{x-1}$. Its domain is the set $(-\infty,1) \cup (1,\infty)$. Since $\dfrac{x^2-1}{x-1} = x+1$, the graph of $y = f(x)$ is the straight line with slope 1 and y-intercept 1, with the point (1,2) removed.



From the graph we can see that the closer x gets to 1, the closer the value of $f(x)$ gets to 2.

In a situation like this, we say the the **limit** of $f(x)$ as x approaches 1 is 2, and **in general:**

we say that the limit as x approaches a is L if the values of $f(x)$ get arbitrarily close to L as long as x is close enough to a.

We write $\lim_{x \to a} f(x) = L$

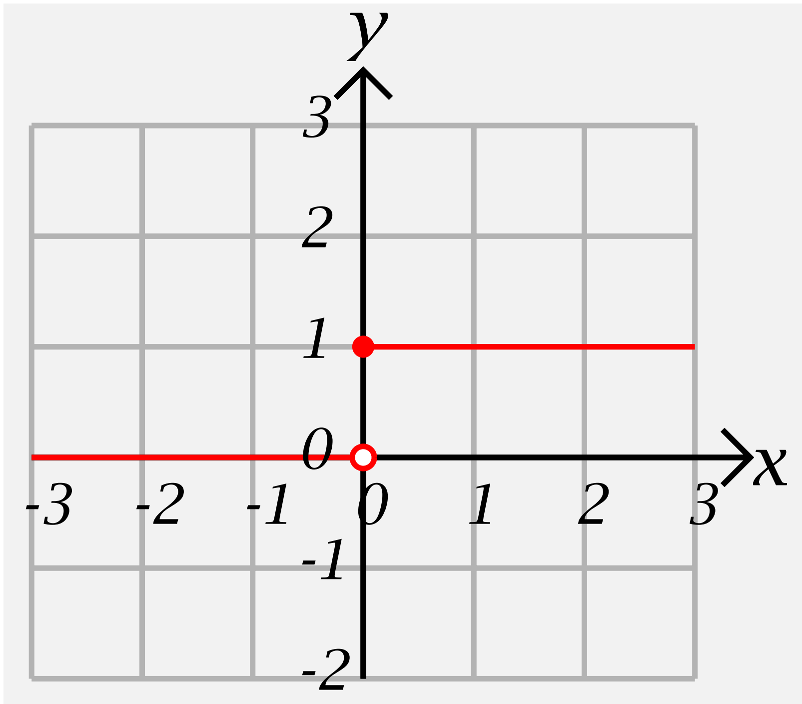
Fact: If $f(x)$ is a polynomial function, a rational function, an exponential function or a logarithmic function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

Thus limits aren't very interesting unless something special is going on with the function, such as a being outside the domain(as in our first example), or a being a point of transition of the function's definition from one formula to another in the case of a function defined with a multiline formula.

Example:

Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Here $a = 0$ is in the domain, but the limit $\lim_{x \to a} f(x)$ does not exist. This function is known as the "Heaviside function"



One-Sided Limits

We also have the concept of One-Sided Limits:

we say that the limit as x approaches a **from the left** is L if the values of $f(x)$ get arbitrarily close to L as long as x is close enough to, **and to the left of a**.

We write $\lim_{x \to a^-} f(x) = L$.

we say that the limit as x approaches a **from the right** is L if the values of $f(x)$ get arbitrarily close to L as long as x is close enough to, **and to the right of a**.

We write $\lim_{x \to a^+} f(x) = L$.

Sometimes we will refer to the ordinary limit $\lim_{x\to a} f(x)$ as a "two-sided limit".

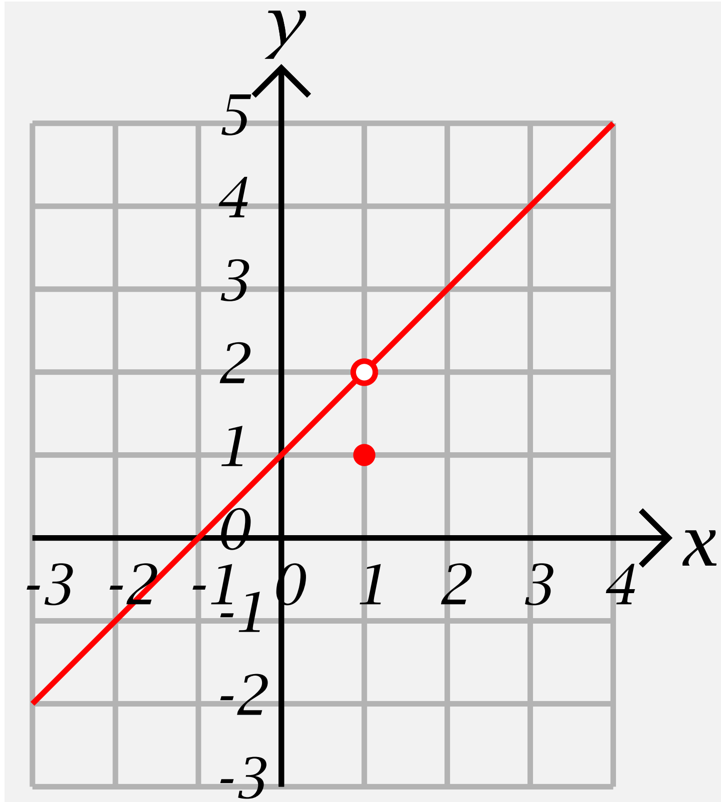
Fact: If $\lim_{x\to a} f(x)$ exists then both $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$ exist. Furthermore, they are both equal to $\lim_{x\to a} f(x)$. Conversely, if both one-sided limits exist **and are equal** , then so does the two-sided limit, and it equals the common value of the two-one-sided lmits.

Example: In the case of the Heaviside function, we have
$\lim_{x\to a} f(x) = 0$ if $a < 0$, $\lim_{x\to a} f(x) = 1$ if $a > 0$, $\lim_{x\to 0^-} f(x) = 0$,
$\lim_{x\to 0^+} f(x) = 1$,
so the two-sided limit exists everywhere except at $a = 0$.

Example: Let

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Here $a = 1$ is in the domain, and the limit $\lim_{x \to a} f(x) = 2$, but $f(1) = 1$, which is not the value of the limit.



Infinite Limits

If the value of a function tends to $+\infty$ or $-\infty$ as x approaches a number a, we will say that the limit is infinite. As before we will have one- and two-sided limits.

Example: Suppose $f(x) = \dfrac{1}{x}$.

Then if $a \neq 0$, we have $\lim_{x \to a} f(x) = \dfrac{1}{a}$, but if $a = 0$, the (two-sided) limit does not exist.

However, we have $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to 0^+} f(x) = +\infty$, and we can also talk about the limits at infinity:
$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$.

Asymptotes

These infinite limits lead us to define **asymptotes** of a function:

If, for a finite number a, $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$, we say that the line $x = a$ is a **vertical asymptote** of f.

If $\lim_{x\to -\infty} f(x) = h$ or $\lim_{x\to \infty} f(x) = k$ for finite numbers h or k, we say that the lines $y = h$ or $y = k$ are **horizontal asymptotes** of f.

There is another type of asymptote not covered in Math 101 called a **slant asymptote:** if $y = mx + b$ and $\lim_{x\to -\infty} |f(x) - (mx + b)| = 0$, then the line $y = mx + b$ is a slant asymptote.