Limit Laws

We let lim stand for any one of the limits $\lim_{x\to a}$, $\lim_{x\to a^-}$, $\lim_{x\to a^+}$, $\lim_{x\to -\infty}$, or $\lim_{x\to +\infty}$, and suppose c is a constant and that $\lim f(x)$ and $\lim g(x)$ exist. Then

(1) $\lim [f(x) + g(x)] = \lim f(x) + \lim g(x)$

(2) $\lim [f(x) - g(x)] = \lim f(x) - \lim g(x)$

(3) $\lim [f(x)g(x)] = \lim f(x) \lim g(x)$

(4) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$ if $\lim g(x)$ is not equal to 0.

(5) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}$, if n is a positive integer and $\lim f(x) \geq 0$ is also required if n is even.

(6) If n is a positive integer, $\lim (f(x))^n = (\lim f(x))^n$

(7) $\lim cf(x) = c \lim f(x)$

(8) If c is a constant then $\lim c = c$ for any a.

The next three limit laws do not apply to limits as $x \to \pm\infty$:

(9) $\lim_{x\to a} x = a$, $\lim_{x\to a^-} x = a$, $\lim_{x\to a^+} x = a$, for any a.

(10) $\lim_{x\to a} x^n = a^n$, $\lim_{x\to a^-} x^n = a^n$, $\lim_{x\to a^+} x^n = a^n$, for any a,
if n is a positive integer.

(11) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, $\lim_{x\to a^-} \sqrt[n]{x} = \sqrt[n]{a}$, $\lim_{x\to a^+} \sqrt[n]{x} = \sqrt[n]{a}$, for any
$a > 0$,
if n is a positive integer.

(12) If $f(x) = g(x)$ for all x except possibly $x = a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ if either limit exists.

(13) If $f(x)$ is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Examples:

Example 1: $\lim_{x \to -1} \left[x^5 - 3x^3 + 1\right](x^2 - 2) =$

$$\left[(-1)^5 - 3(-1)^3 + 1\right]((-1)^2 - 2) =$$

$$[-1 - 3(-1) + 1](1 - 2) = [-1 + 3 + 1](-1) = (3)(-1) = \text{-3}$$

Example 2:

$$\lim_{x \to 25} \frac{\sqrt{x}}{x + 25} = \frac{\sqrt{25}}{25 + 25} = \frac{5}{50} = \frac{1}{10}$$

Example 3:

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right)\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right) =$$

$$\lim_{x\to 9}\frac{(\sqrt{x})^2-3^2}{(x-9)(\sqrt{x}+3)} =$$

$$\lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{\lim_{x\to 9}(\sqrt{x}+3)} =$$

$$\frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

Example 4:

$$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Example 5:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} (x+1) = 1 + 1 = 2$$

Example 6: $\lim_{x\to 0} |x|$

DANGER! $|x|$ is a function defined by a multicolumn formula, and different formulas apply for $|x|$ on either side of the limiting value of x, namely 0. We must calculate the two one-sided limits:

$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = (-0) = 0$$

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$$

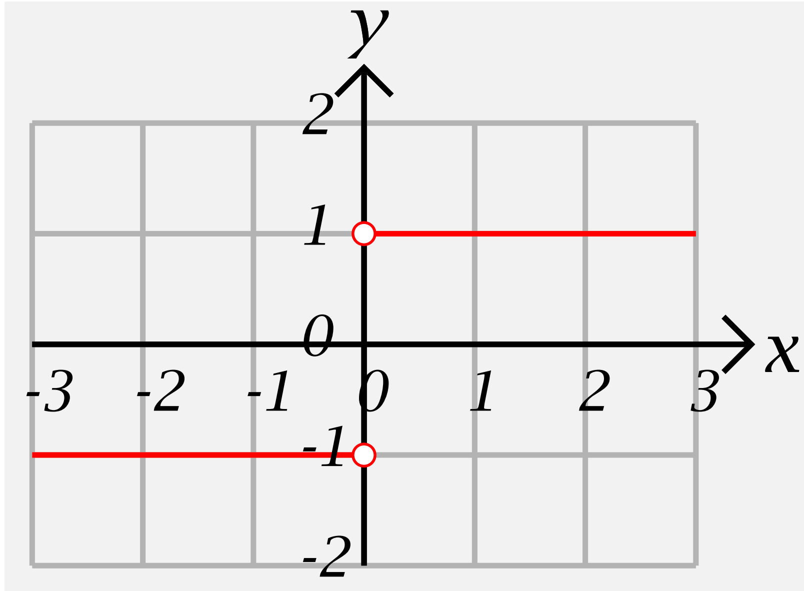
Since the left and right hand limits are equal, we can conclude that the two-sided limit does exist and equals the common value 0.

Example 7:

$$\lim_{x \to 0} \frac{|x|}{x}$$

Note that the limiting value of x, 0, is not in the domain of the function $\frac{|x|}{x}$.

Again, $|x|$ is a function defined by a multicolumn formula, and different formulas apply for x on either side of the limiting value of x, so again we must calculate the two one-sided limits:



$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

Since the left and right hand limits are unequal, we can conclude that the two-sided limit does not exist.

Limits of Rational Functions

If $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to a} f(x) = f(a) = \frac{p(a)}{q(a)} \quad \textbf{if } q(a) \neq 0$$

If $q(a) = 0$, then there are a number of possibilities:

if $p(a) \neq 0$, we could have $\lim_{x \to a} f(x) = +\infty$, or $\lim_{x \to a} f(x) = -\infty$,
or
$\lim_{x \to a-} f(x) = -\infty$ and $\lim_{x \to a+} f(x) = +\infty$, or $\lim_{x \to a-} f(x) = +\infty$ and $\lim_{x \to a+} f(x) = -\infty$.

If $p(a) = 0$, it is possible, but not guaranteed, that $\lim_{x \to a} f(x)$ can exist.

Example 8: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Here $a = 2$, $p(x) = x^2 - 4$, $q(x) = x - 2$, so $p(a) = p(2) = 0$, $q(a) = q(2) = 0$. We have to do some algebraic manipulation to compute the limit:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} x + 2 = 2 + 2 = 4.$$

Rational Functions at $\pm\infty$

Theorem: If $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then:

(1) if the degree of $p(x)$ is less than the degree of $q(x)$,
$\lim_{x\to\pm\infty} f(x) = 0$,

(2) if the degree of $p(x)$ is greater than the degree of $q(x)$,
$\lim_{x\to\pm\infty} f(x) = \pm\infty$,

(3) if the degree of $p(x)$ equals the degree of $q(x)$,
$\lim_{x\to\pm\infty} f(x) = \dfrac{a_0}{b_0}$,
where a_0 and b_0 are the leading coefficients of $p(x)$ and $q(x)$.

Example 9: The **Floor and Ceiling Functions**.

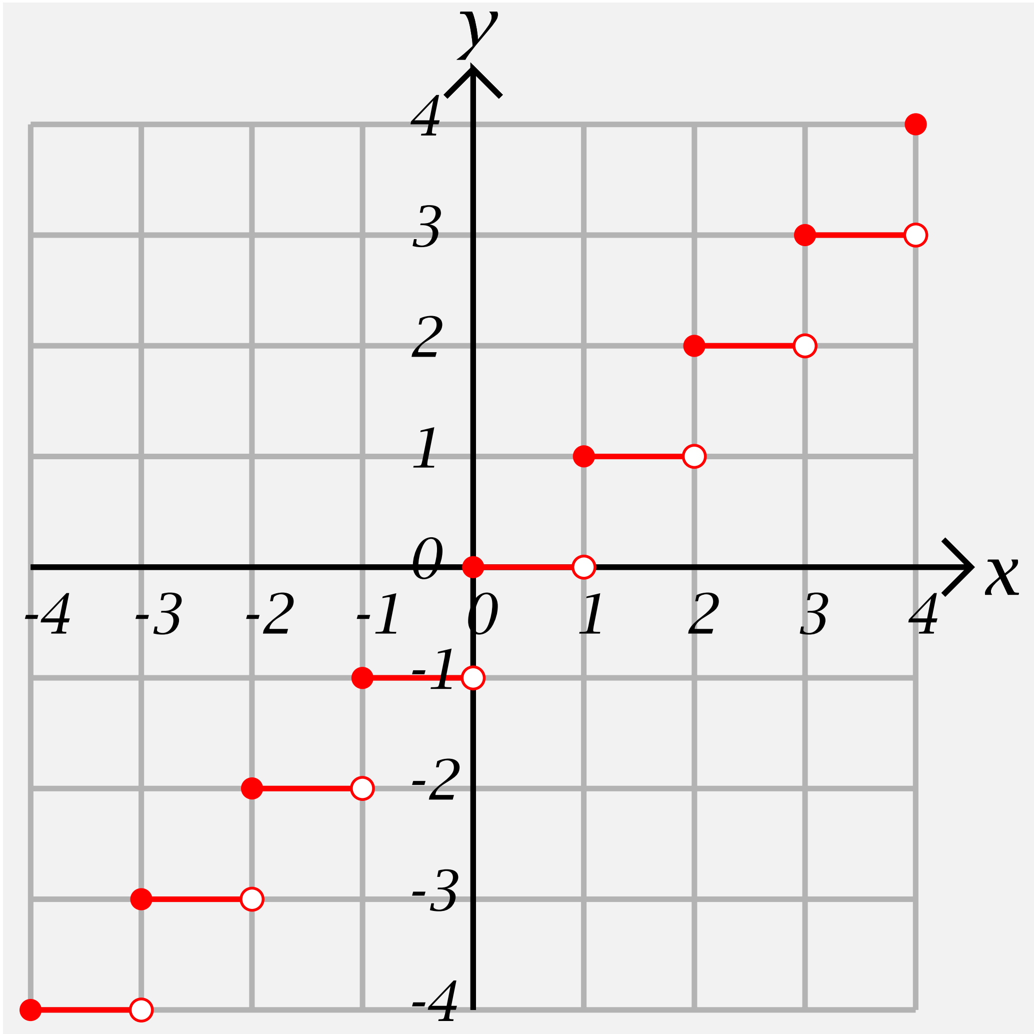
We define $\lfloor x \rfloor$, the floor of x, to be the largest integer that is less than or equal to x, and $\lceil x \rceil$, the ceiling of x, to be the least integer that is greater than or equal to x. In other words, the floor function rounds x down to the nearest integer and the ceiling function rounds x up to the nearest integer. They are both examples of step functions. They both have two-sided limits at non-integer values:

If a is not an integer, then $\lim_{x\to a} \lfloor x \rfloor = \lfloor a \rfloor$, and $\lim_{x\to a} \lceil x \rceil = \lceil a \rceil$, but if a is an integer, things are different:

$$\lim_{x\to a-} \lfloor x \rfloor = \lfloor a \rfloor - 1, \qquad \lim_{x\to a+} \lfloor x \rfloor = \lfloor a \rfloor$$

and $$\lim_{x\to a-} \lceil x \rceil = \lceil a \rceil, \qquad \lim_{x\to a+} \lceil x \rceil = \lceil a \rceil + 1$$

The Floor Function:



The Ceiling Function: