Continuity

Continuity at a point

Definition: A function f is continuous at a point $x = a$ in its domain if

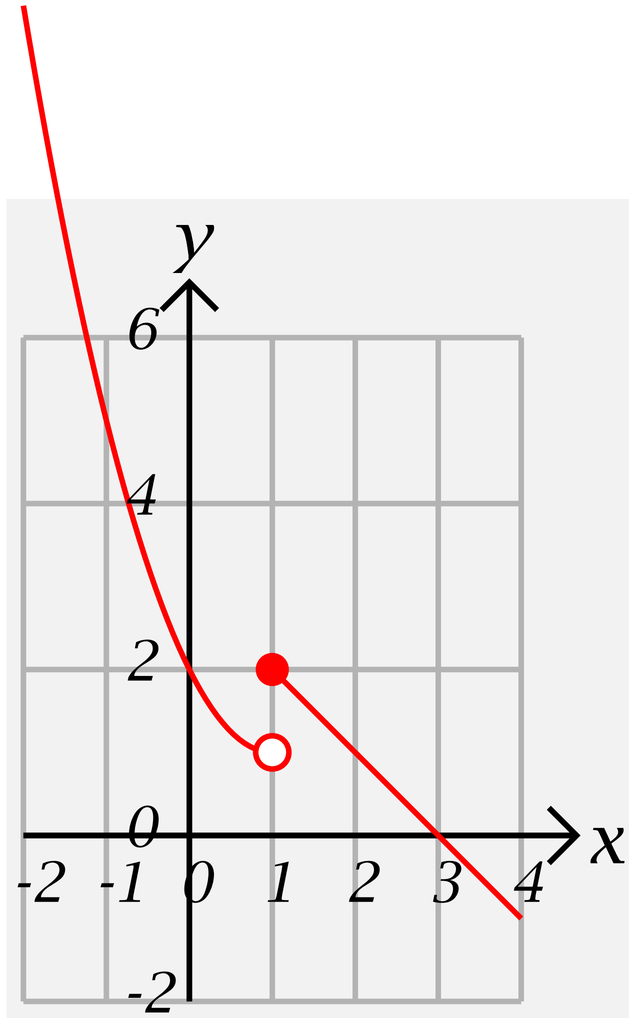
$$\lim_{x \to a} f(x) \text{ exists and equals } f(a).$$

Discontinuities

f is said to be **discontinuous** at any number where it is not continuous. If $\lim_{x \to a} f(x)$ exists but does not equal $f(a)$ we say that the discontinuity is **removable**. If both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (i.e. are finite numbers) but are unequal we say that the discontinuity is a **jump discontinuity**. Otherwise the discontinuity is said to be an **infinite discontinuity**.

Example 1: Let $f(x) = \begin{cases} (x-1)^2+1 & \text{if } x < 1 \\ 3-x & \text{if } 1 \le x \end{cases}$

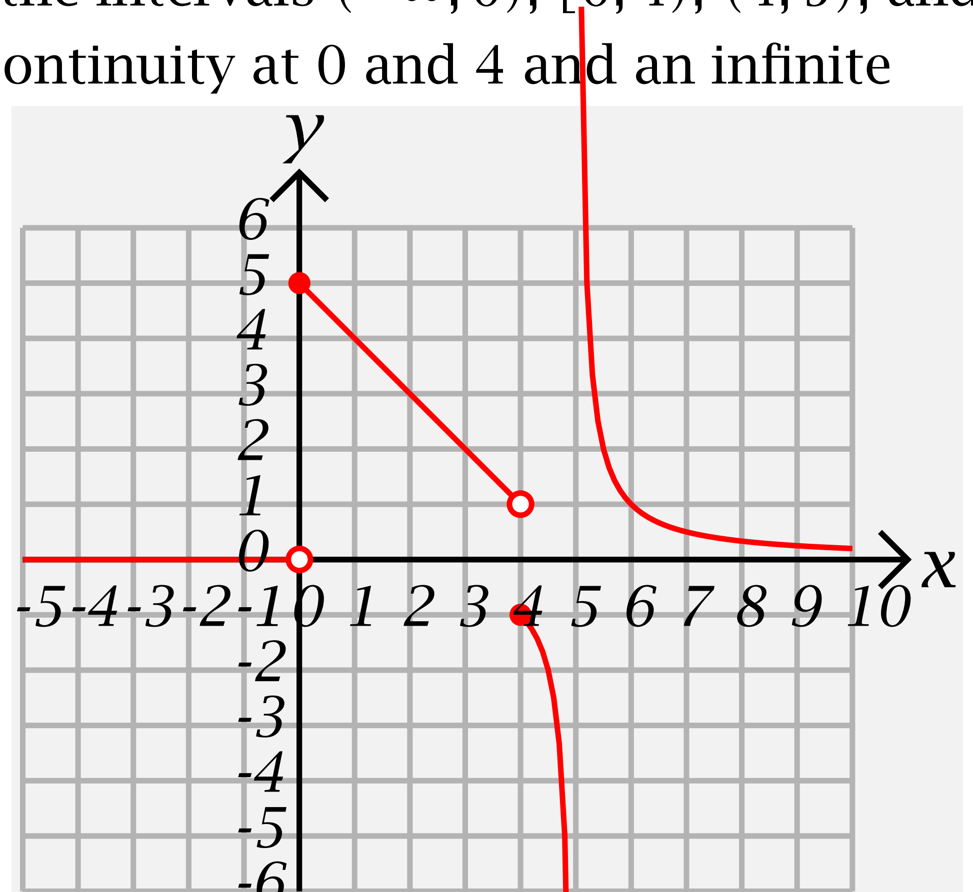
Then f is continuous on the intervals $(-\infty, 1)$ and $[1, \infty)$, but it is **NOT** continuous at 1. There is a jump discontinuity at 1.



Example 2:

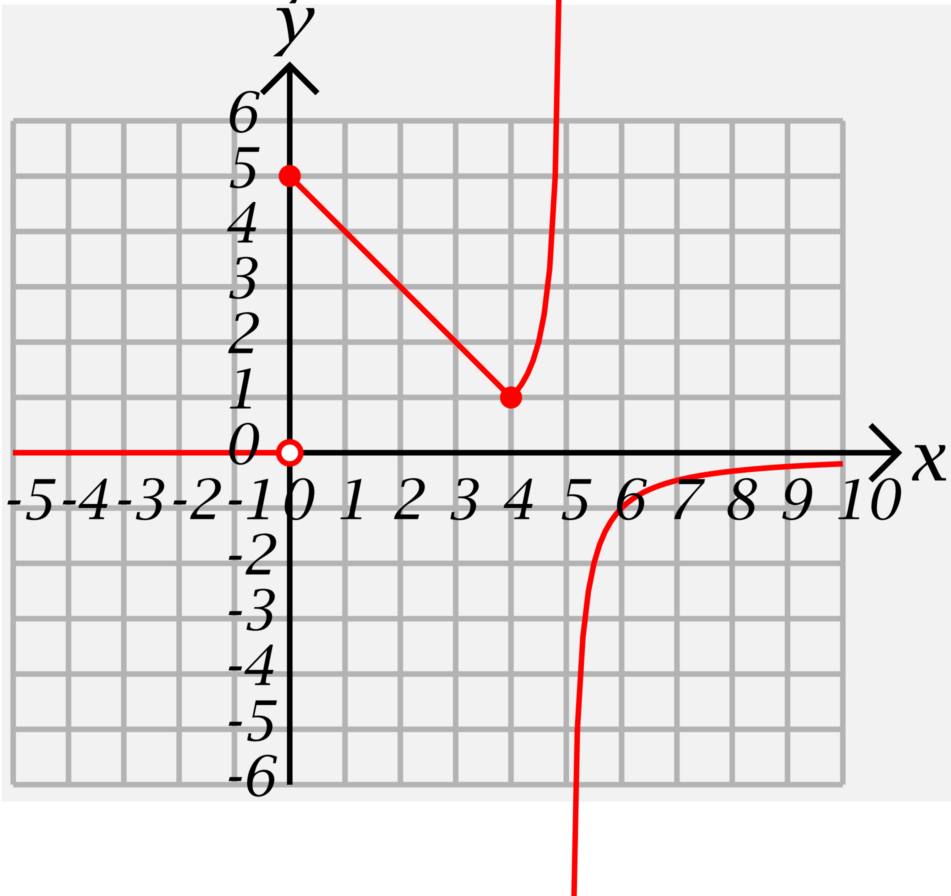
$$\text{Let } f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 5 - x & \text{if } 0 \le x < 4 \\ \frac{1}{x-5} & \text{if } 4 \le x \text{ and } x \ne 5 \end{cases}$$

Then f is continuous on the intervals $(-\infty, 0)$, $[0, 4)$, $(4, 5)$, and $(5, \infty)$. It has a jump discontinuity at 0 and 4 and an infinite discontinuity at 5.



Example 3: Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 5 - x & \text{if } 0 \leq x < 4 \\ \frac{-1}{x-5} & \text{if } 4 \leq x \text{ and } x \neq 5 \end{cases}$

Then f is continuous on the intervals $(-\infty, 0)$, $(0, 5)$, and $(5, \infty)$. It has a jump discontinuity at 0 and an infinite discontinuity at 5.



Definition: A function f is continuous from the left at a number a in its domain if

$\lim_{x \to a^-} f(x)$ exists and equals $f(a)$.

It is continuous from the right at a number a in its domain if

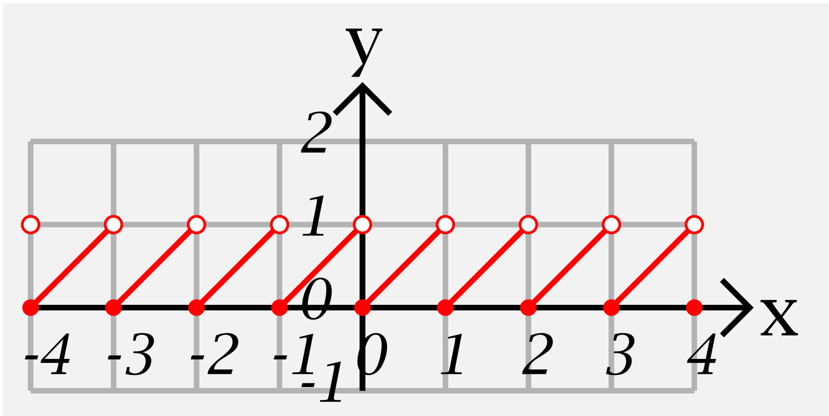
$\lim_{x \to a^+} f(x)$ exists and equals $f(a)$.

Definition: A function f is continuous on an open interval (c, d) if it is continuous at every number a in the interval (c, d).

Definition: A function f is continuous on a closed interval $[c, d]$ if it is continuous at every number a in the interval (c, d) and continous from the right at c and continuous from the left at d.

Definition: A function f is continuous on a set E if it is continuous on every open interval contained in E.

Example 4: Let $f(x) = x - \lfloor x \rfloor$
Then f is continuous on $(-\infty, \infty) R - Z$, where Z denotes the set of integers. Every integer is a jump discontinuity.



Theorem: If the functions f and g are continuous at a and if c is a constant, then the functions $f+g$, $f-g$, cf, fg, and

$\frac{f}{g}$(if $g(a) \neq 0)$

are also continuous at a.

Theorem: Polynomial, rational, root, exponential and logarithmic functions, and algebraic combinations of such functions are continuous on their domains.

Theorem: If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$ then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b)$$

Theorem: Compositions of continous functions are continuous on their domains.

Intermediate Value Theorem(IVT): If f is continuous on $[a, b]$ then f takes on every value between $f(a)$ and $f(b)$. In other words, every horizontal line lying between the horizontal lines $y = f(a)$ and $y = f(b)$ intersects the graph of f in at least one point.