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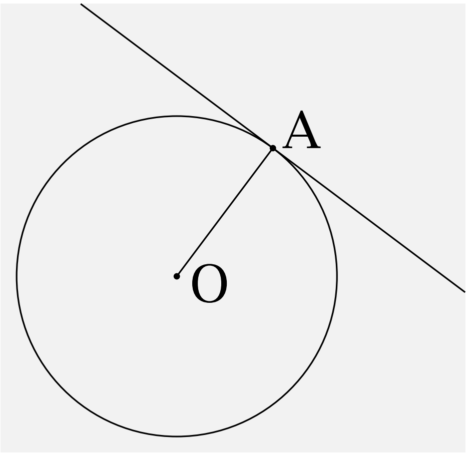


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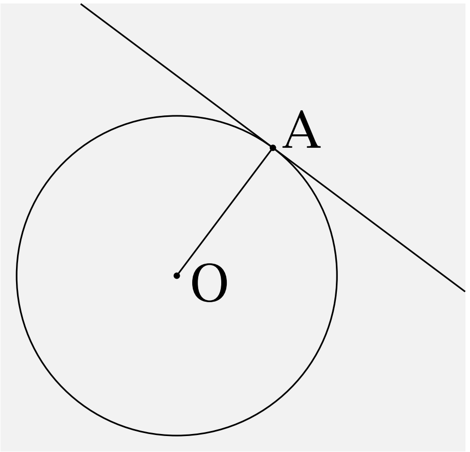


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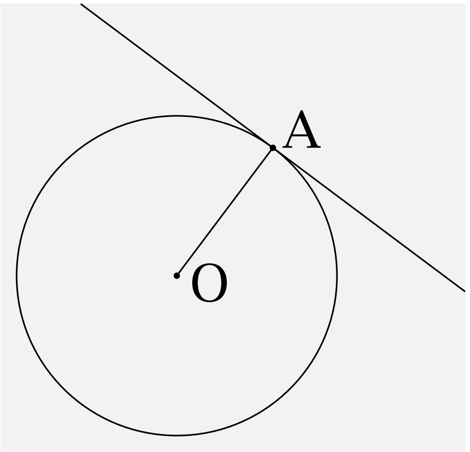


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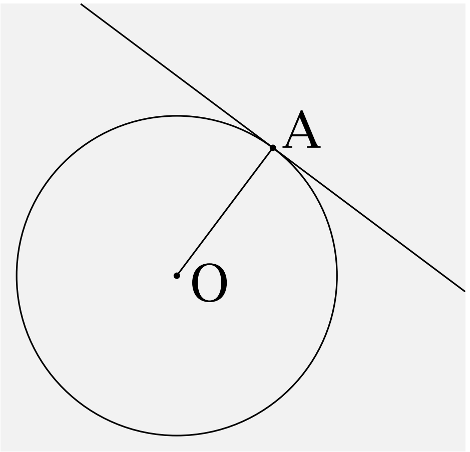


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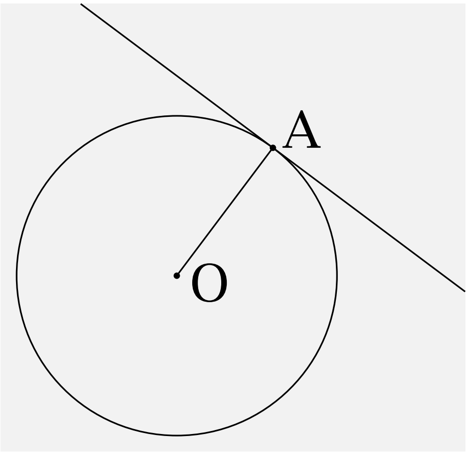


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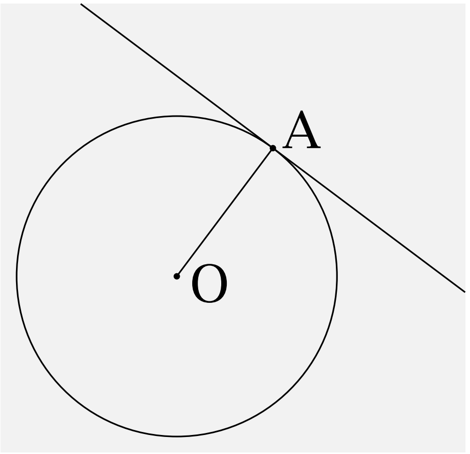


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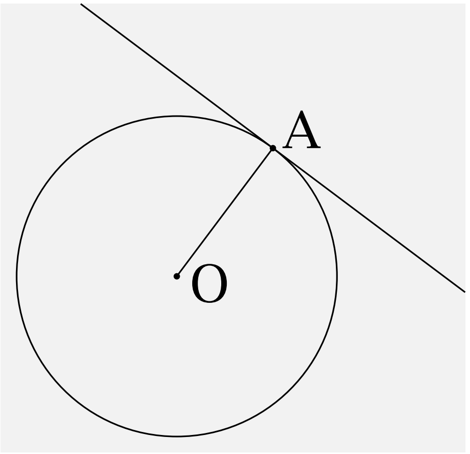


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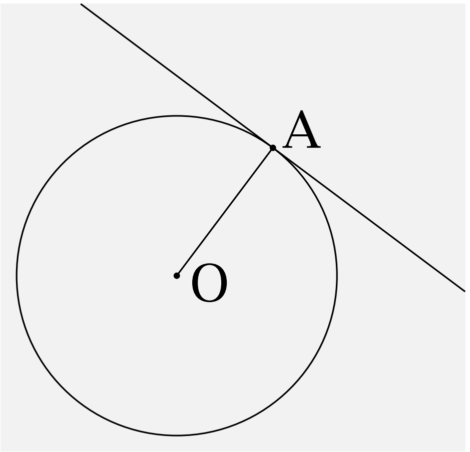


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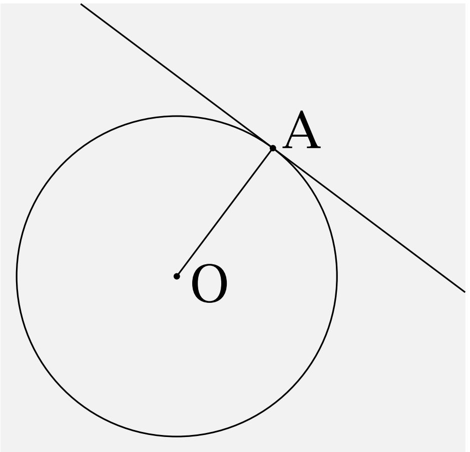


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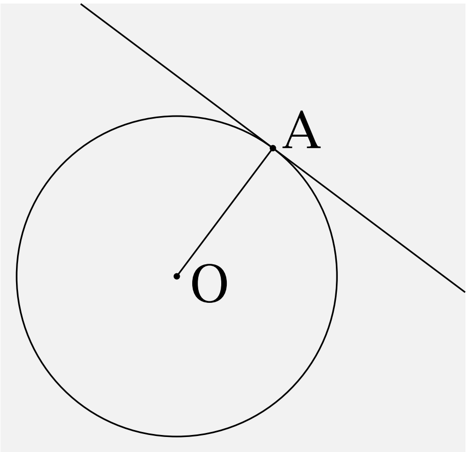


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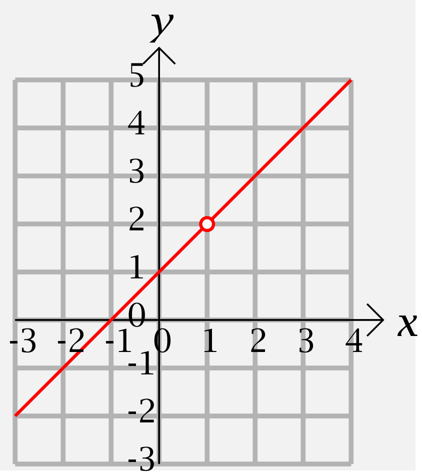


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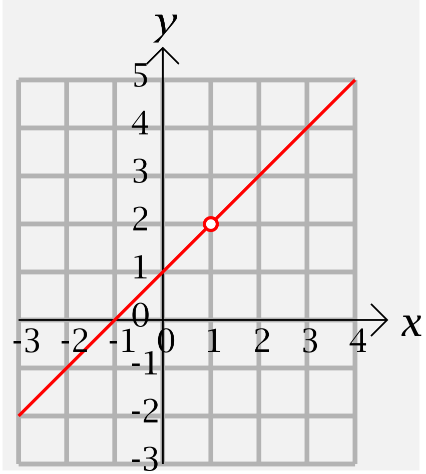


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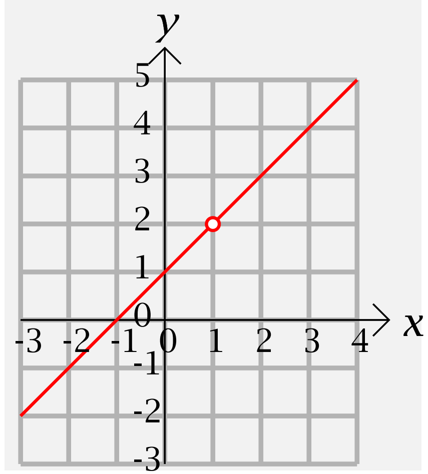


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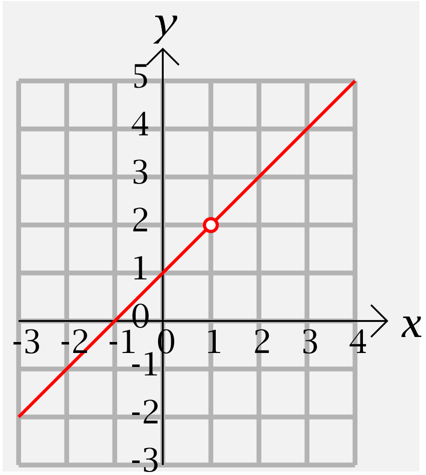


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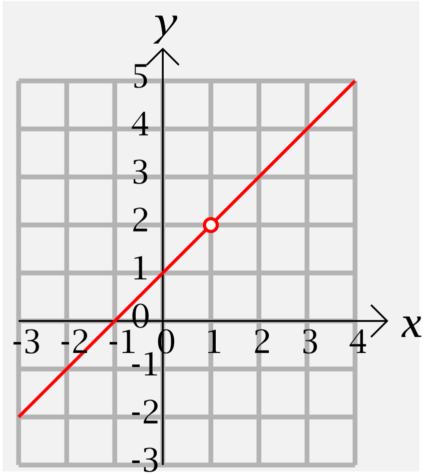


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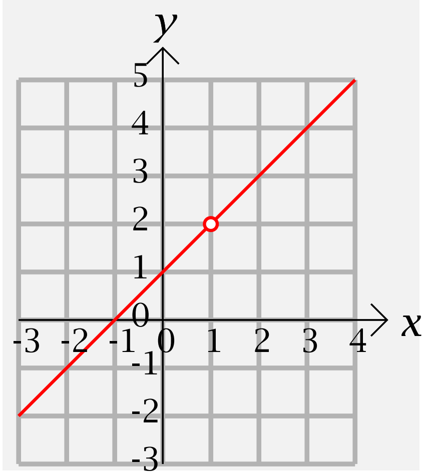


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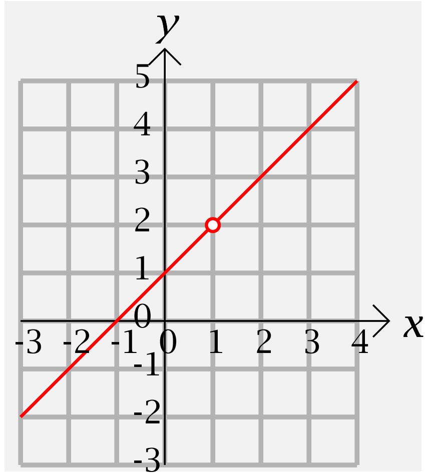


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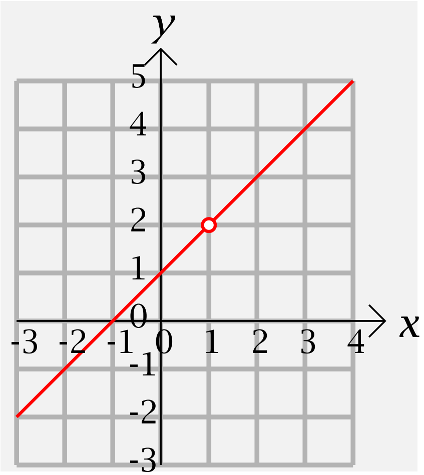


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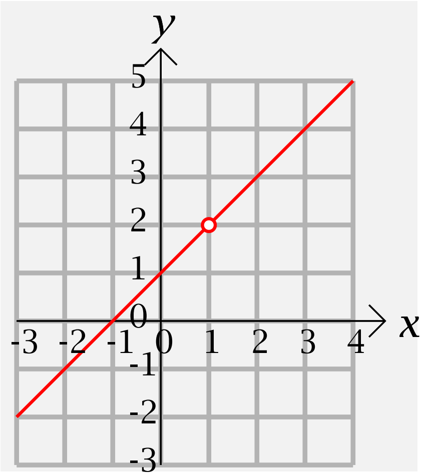


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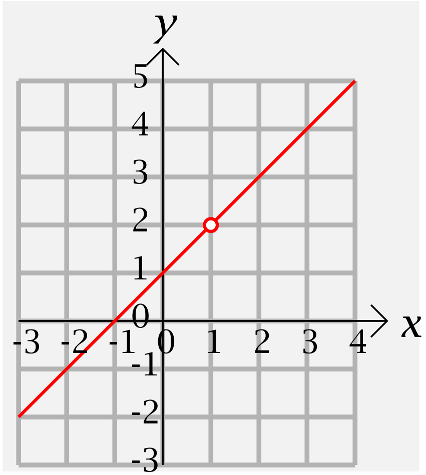


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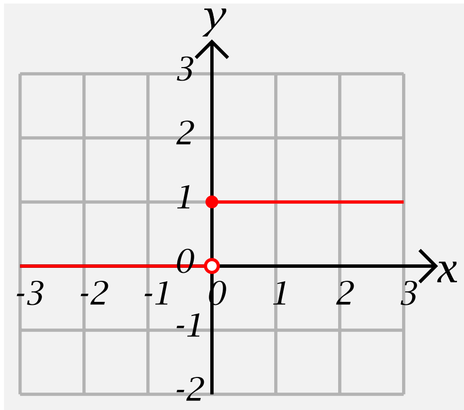


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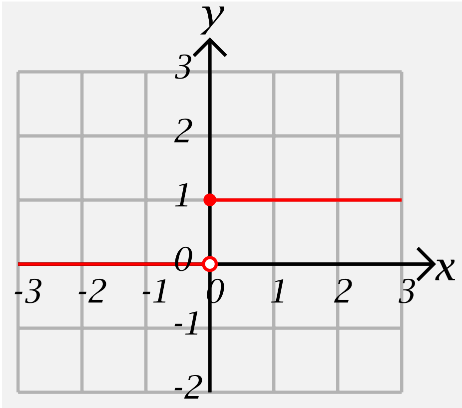


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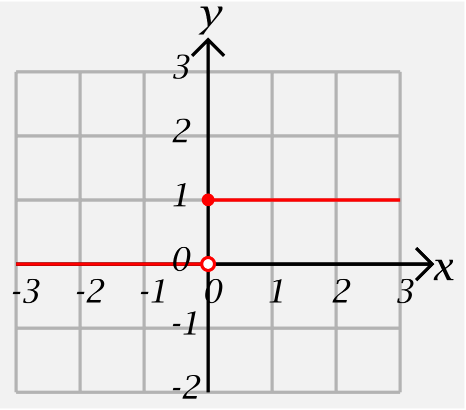


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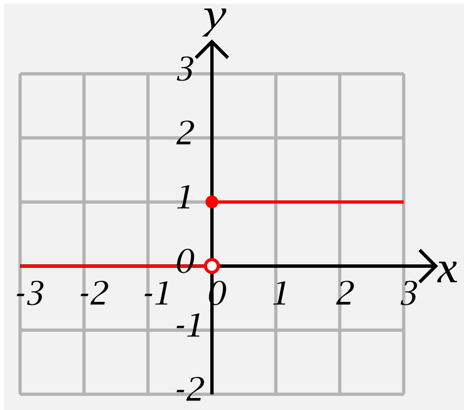


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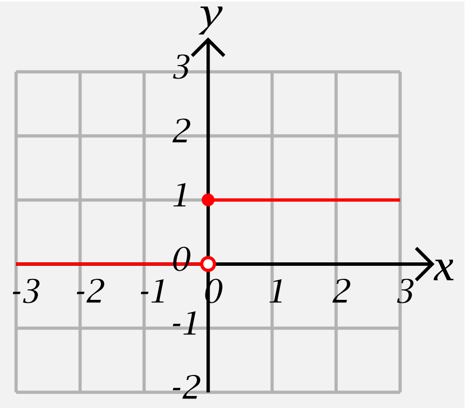


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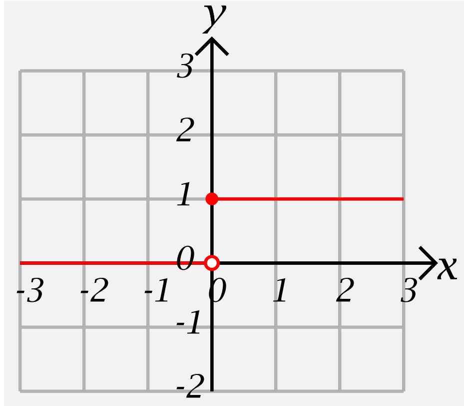


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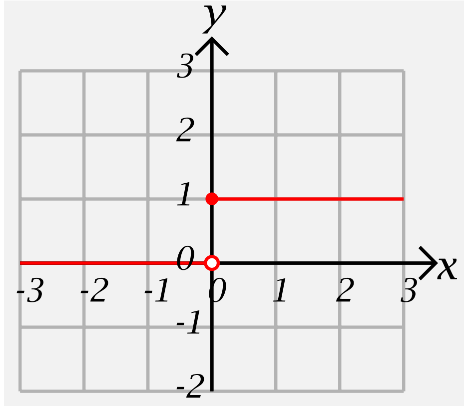


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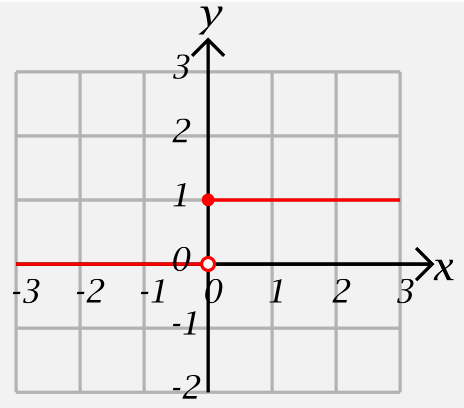


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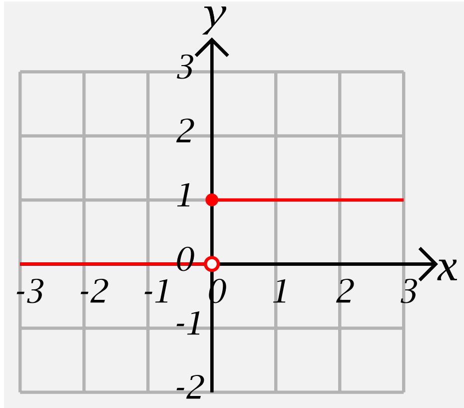


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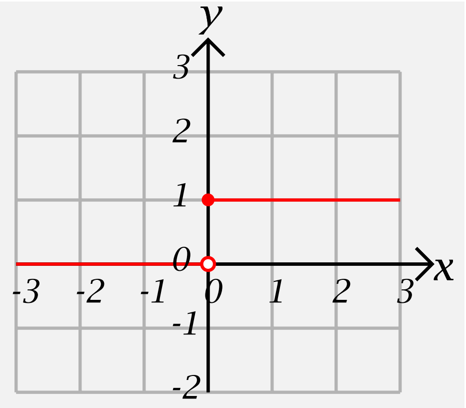


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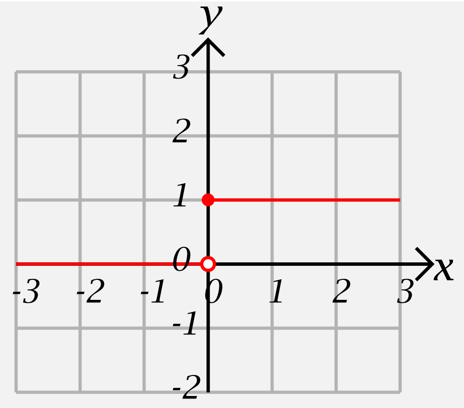


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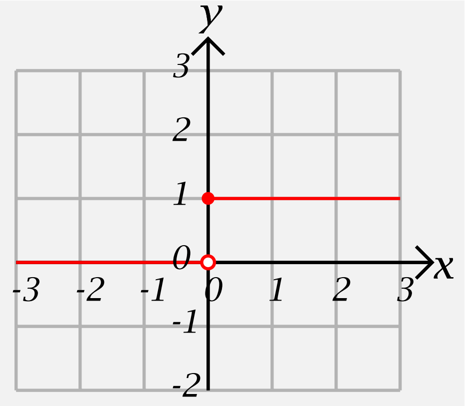


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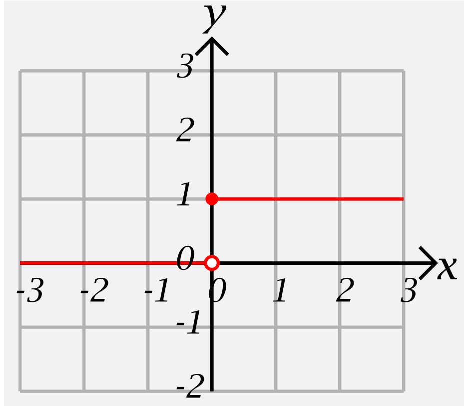


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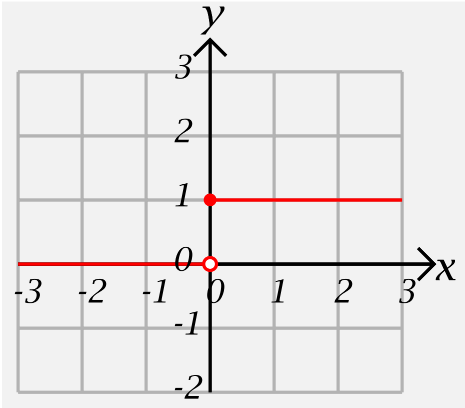
**Fact:** If $\lim_{x \to a} f(x)$ exists then both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$ exist. Furthermore, they are both equal to $\lim_{x \to a} f(x)$.

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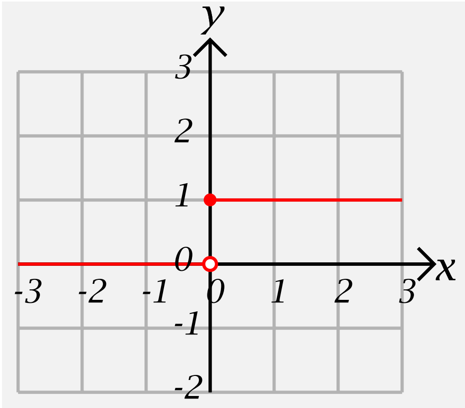


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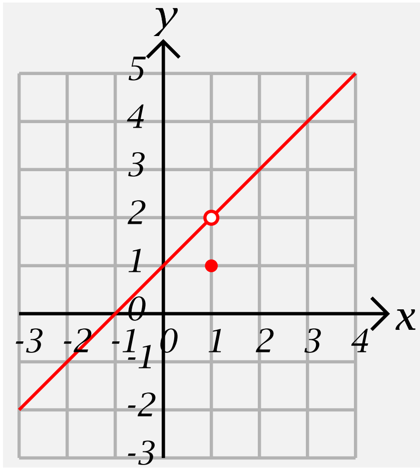


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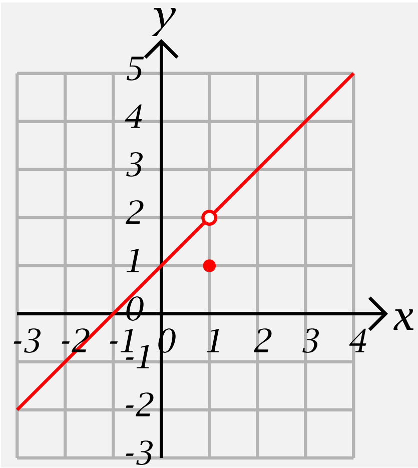


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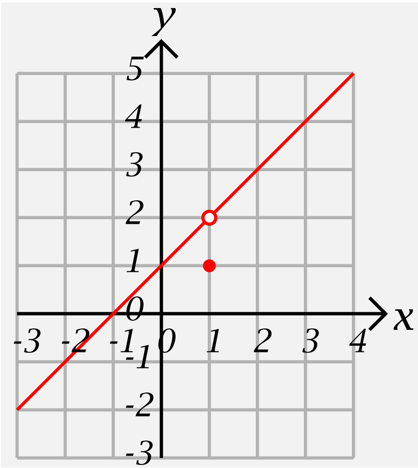


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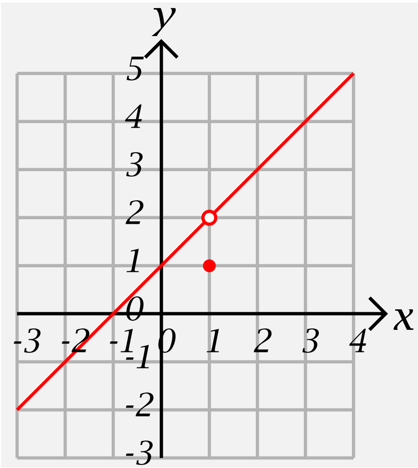


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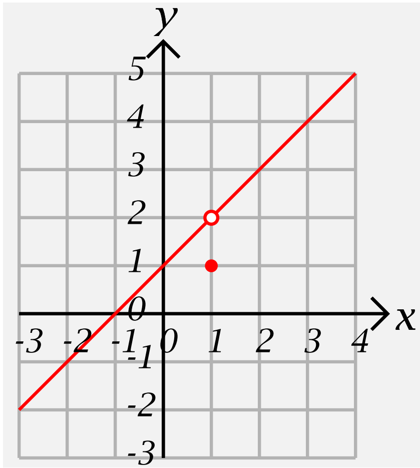


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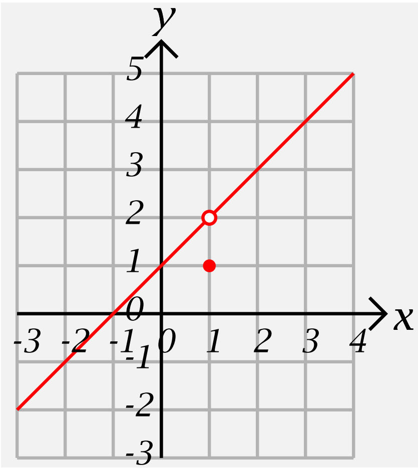


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There is another type of asymptote not covered in Math 101 called a slant asymptote: if $y = mx + b$ and $\lim_{x\to -\infty} |f(x) - (mx + b)| = 0$, then the line $y = mx + b$ is a slant asymptote.