

New Functions From Old Ones

Arithmetic Combinations of Functions

Two functions f and g with domains D and E can be combined arithmetically as

$$f(x) + g(x), \quad f(x) - g(x), \quad f(x)g(x), \quad \text{and} \quad \frac{f(x)}{g(x)}.$$

The domains of the first three will be $D \cap E$, where $D \cap E$ denotes the **intersection** of the sets D and E :

the set of all numbers which belong to **both** sets,

while that of the last function will be $D \cap E$ less all values of x for which $g(x) = 0$.

Students may practice **graphical addition** with a [Java applet](#) .

Transformations of Functions

Vertical & Horizontal Shifts

Suppose the graph of a function $y = f(x)$ is given.

If c is a positive number there are a number of simple ways we may modify the function. It is fairly clear that $y = f(x) + c$ will be the graph shifted **up** a distance c , and $y = f(x) - c$ will be the graph shifted **down** a distance c . In other words, operating on the function **outside** the brackets causes a **vertical** shift.

What happens if we perform the same operations **inside** the brackets?

In other words, how are the graphs of $y = f(x + c)$ and $y = f(x - c)$ related to the graph $y = f(x)$? The easy way to figure this out is to let $x = 0$. Then $y = f(0 + c) = f(c)$, so the value of $f(x)$ at c is shifted to be the value of f at 0. Thus the graph is shifted a distance c to the **left**. Similarly, the graph of $y = f(x - c)$ is that of $y = f(x)$ shifted a distance c to the **right**. In other words, operating on the function **inside** the brackets causes a **horizontal** shift.

Students may practice graph shifting with a **Java applet**.

Stretching & Reflecting Functions

If $c > 1$, there are some more simple ways we may modify the function. It is fairly clear that $y = cf(x)$ will be the graph **stretched** or **magnified** **vertically** by the factor c , and $y = \frac{1}{c}f(x)$ will be the graph **compressed** or **shrunk** **vertically** by the factor c . What happens if we perform the same operations **inside** the brackets?

In other words, how are the graphs of $y = f(cx)$ and $y = f(\frac{1}{c}x)$ related to the graph $y = f(x)$? The easy way to figure this out is to let $x = 1$. Then $y = f(c(1)) = f(c)$, so the value of $f(x)$ at c is shifted to be the value at 1. Thus the graph is scaled by the factor c **horizontally**.

Similarly, the graph of $y = f\left(\frac{1}{c}x\right)$ is that of $y = f(x)$ scaled by the factor $\frac{1}{c}$ **horizontally**.

What happens if $c < 0$? To make things simple, we assume $c = -1$. Then the graph of $y = -f(x)$ is just that of $y = f(x)$ **reflected** about the x -axis, a **vertical** effect, while the graph of $y = f(-x)$ is that of $y = f(x)$ **reflected** about the y -axis, a **horizontal** effect.

Remember:

Operations **outside** the brackets have a **vertical** effect, while those **inside** the brackets have a **horizontal** effect.

Composition of Functions

Whenever we take a function of a function, we say we are **composing** them.

Example 1: Suppose $f(x) = x^2$ and $g(x) = x + 1$.

Then $f(g(x)) = (g(x))^2 = (x + 1)^2 = x^2 + 2x + 1$ is denoted by $f \circ g(x)$, so that we have

$$f \circ g(x) = f(g(x)) = (g(x))^2 = (x + 1)^2 = x^2 + 2x + 1.$$

Note that **order is important:**

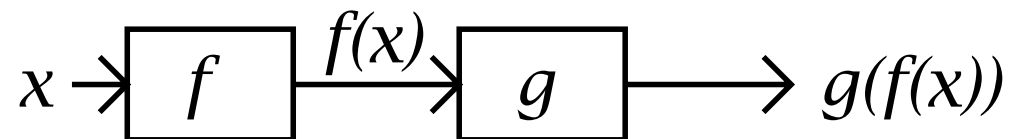
$$g \circ f(x) = g(f(x)) = (f(x)) + 1 = x^2 + 1 \text{ is quite different!}$$

Definition: the **composition** of two functions f and g is defined to be the function $f \circ g$ defined by the rule

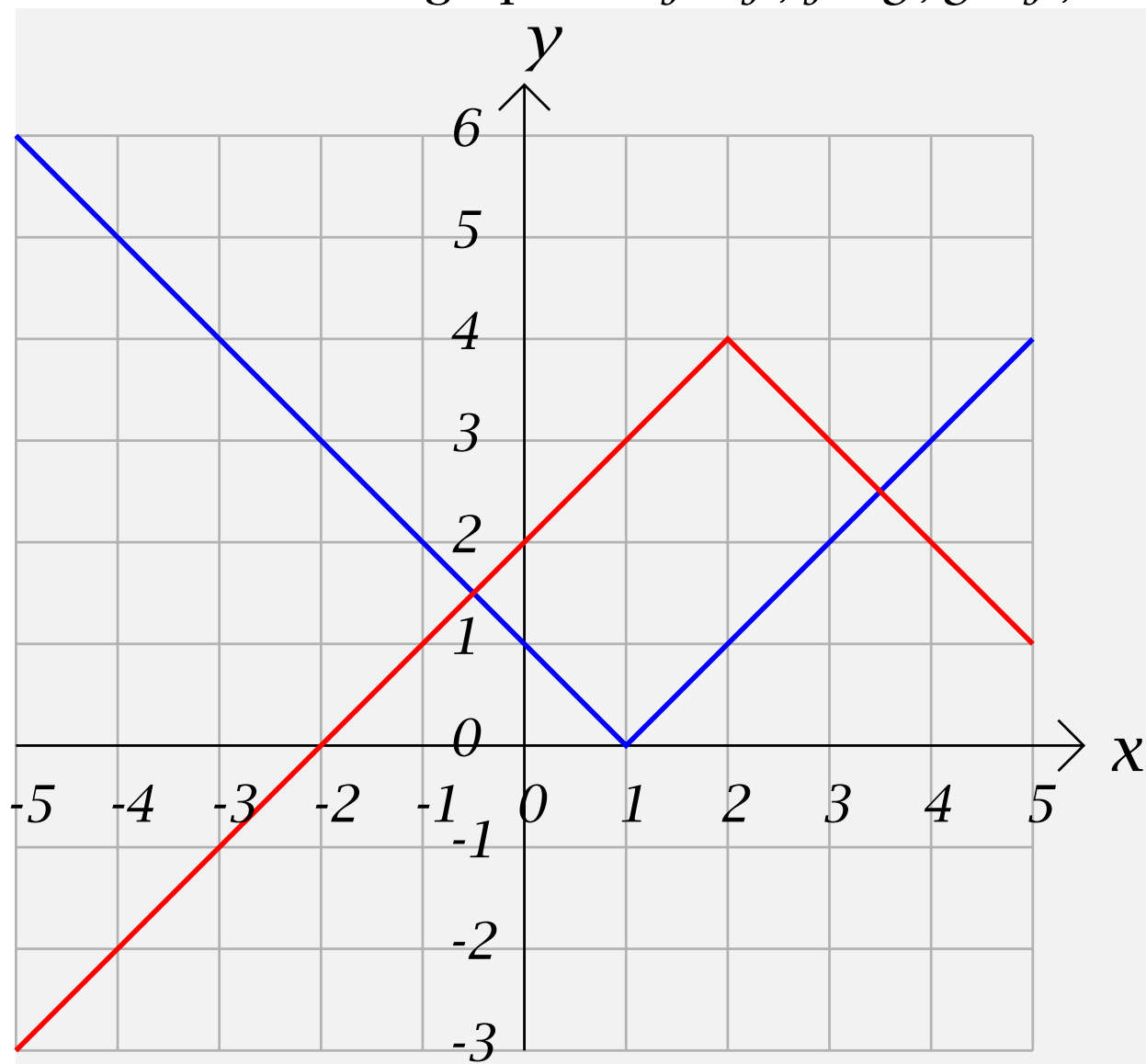
$$f \circ g(x) = f(g(x))$$

Students may practice **graphical composition** with a **Java applet** .

It is convenient to think of functions in this context as processing machines connected in series.



Example The graph of f is shown in blue, and the graph of g is shown in red. Draw the graphs of $f \circ f$, $f \circ g$, $g \circ f$, and $g \circ g$.



It is always a good idea to make up a table of values for the functions to be graphed:

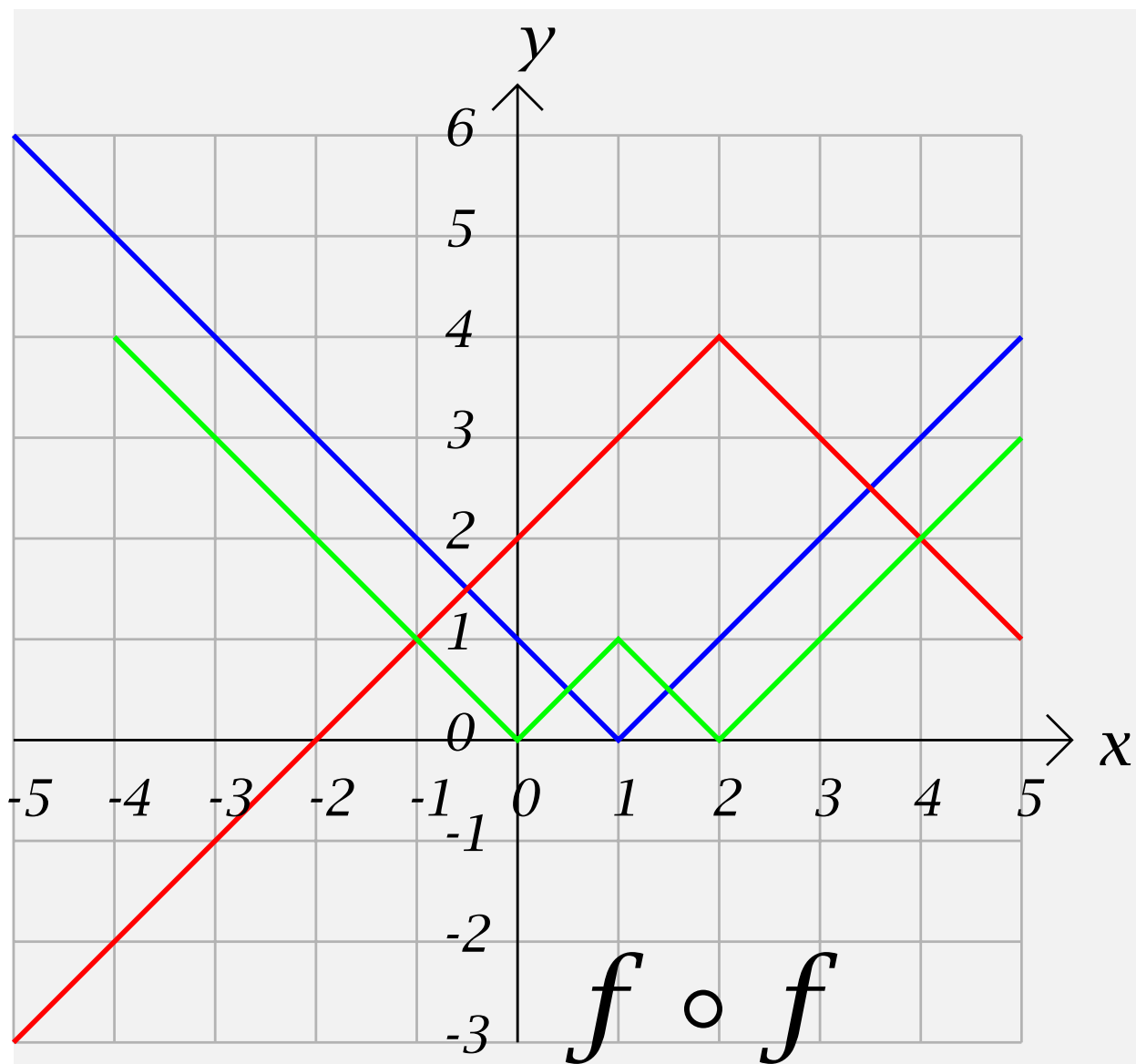
x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f(x)$	6	5	4	3	2	1	0	1	2	3	4
$g(x)$	-3	-2	-1	0	1	2	3	4	3	2	1
$f(f(x))$	U	4	3	2	1	0	1	0	1	2	3
$f(g(x))$	4	3	2	1	0	1	2	3	2	1	0
$g(f(x))$	U	1	2	3	4	3	2	3	4	3	2
$g(g(x))$	-1	0	1	2	3	4	3	2	3	4	3

, where

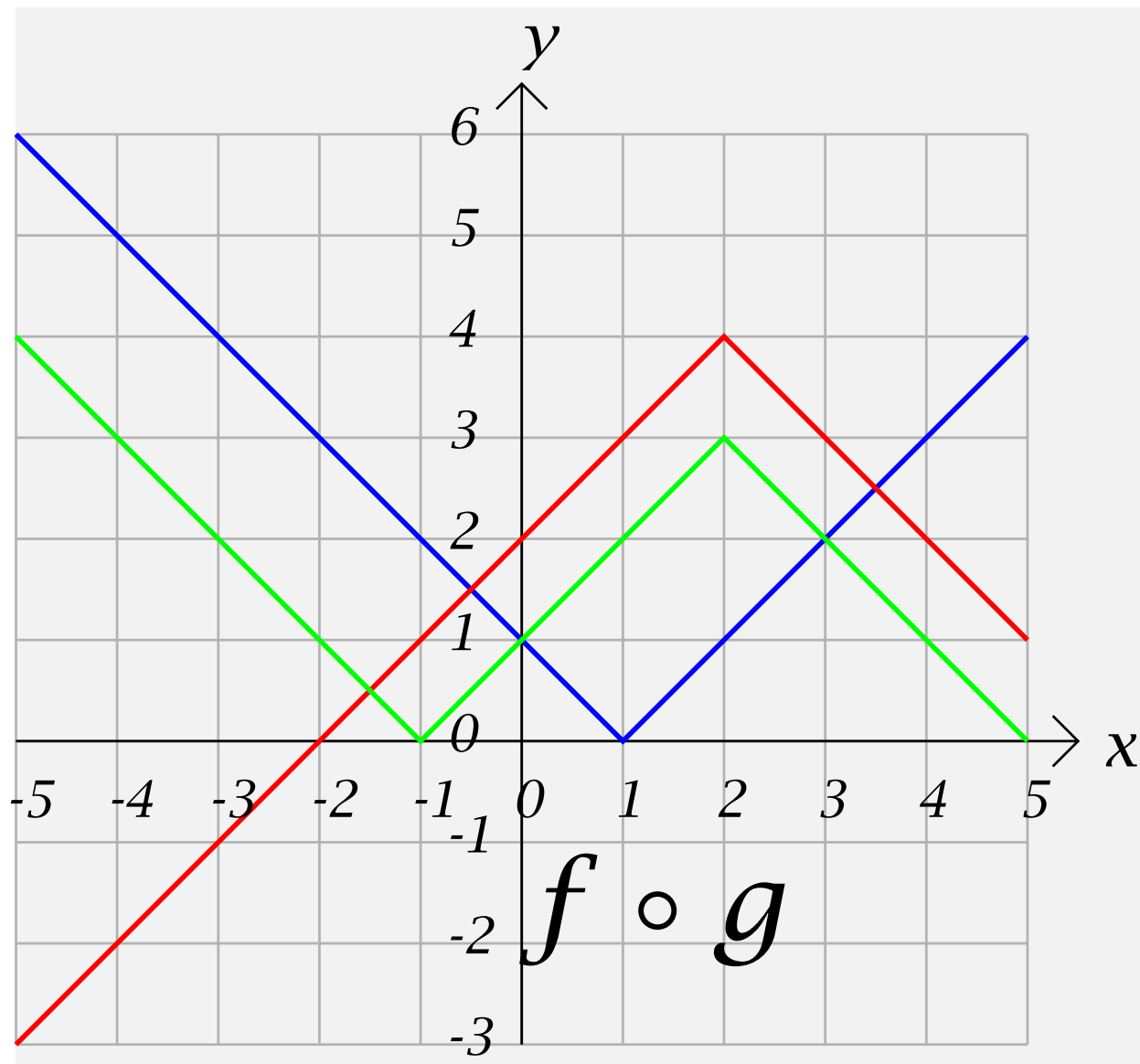
U = "Undefined".

The graphs of all four composite functions are made up of straight line segments. Why? They are shown in green:

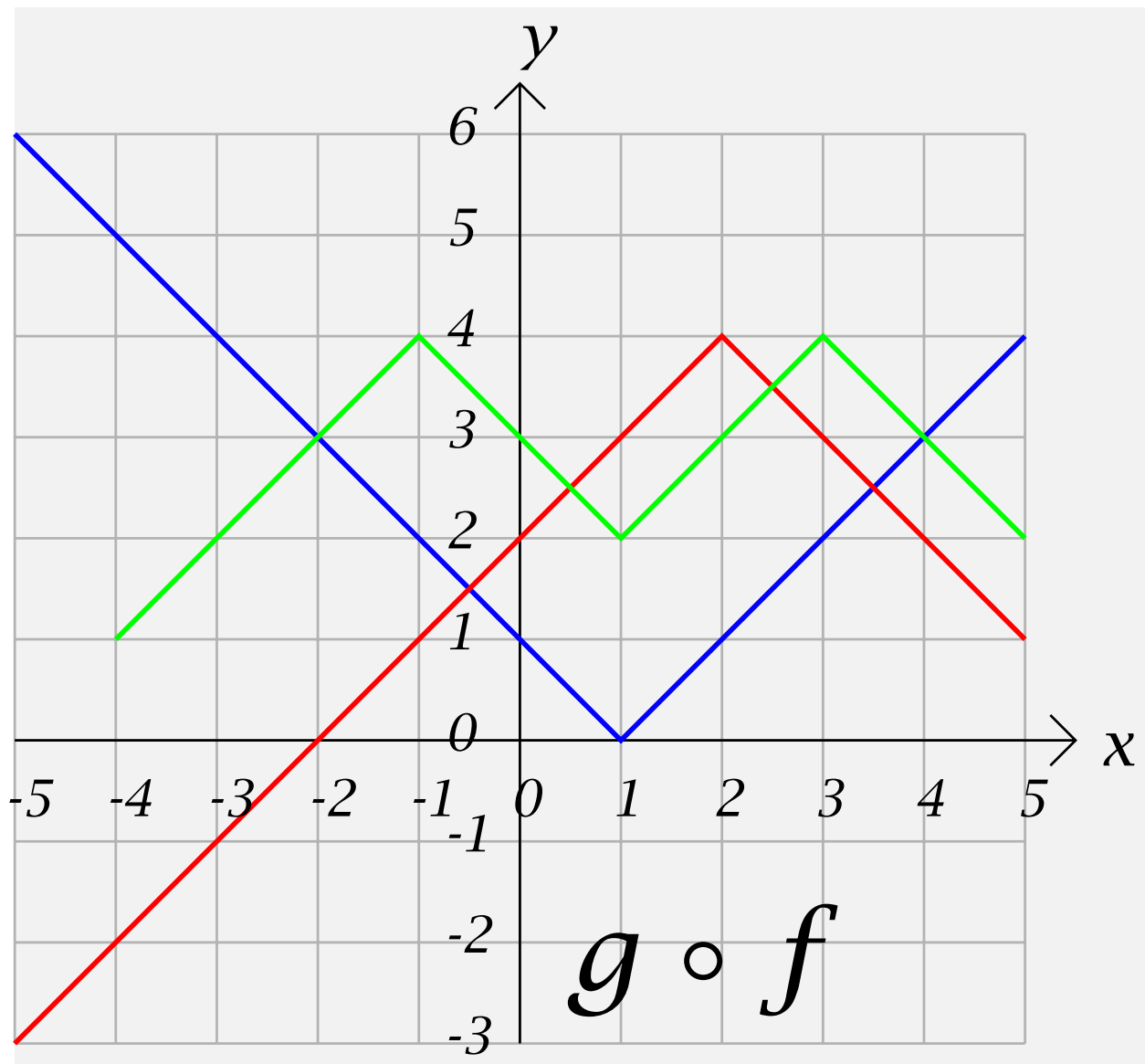
New Functions From Old Ones-10



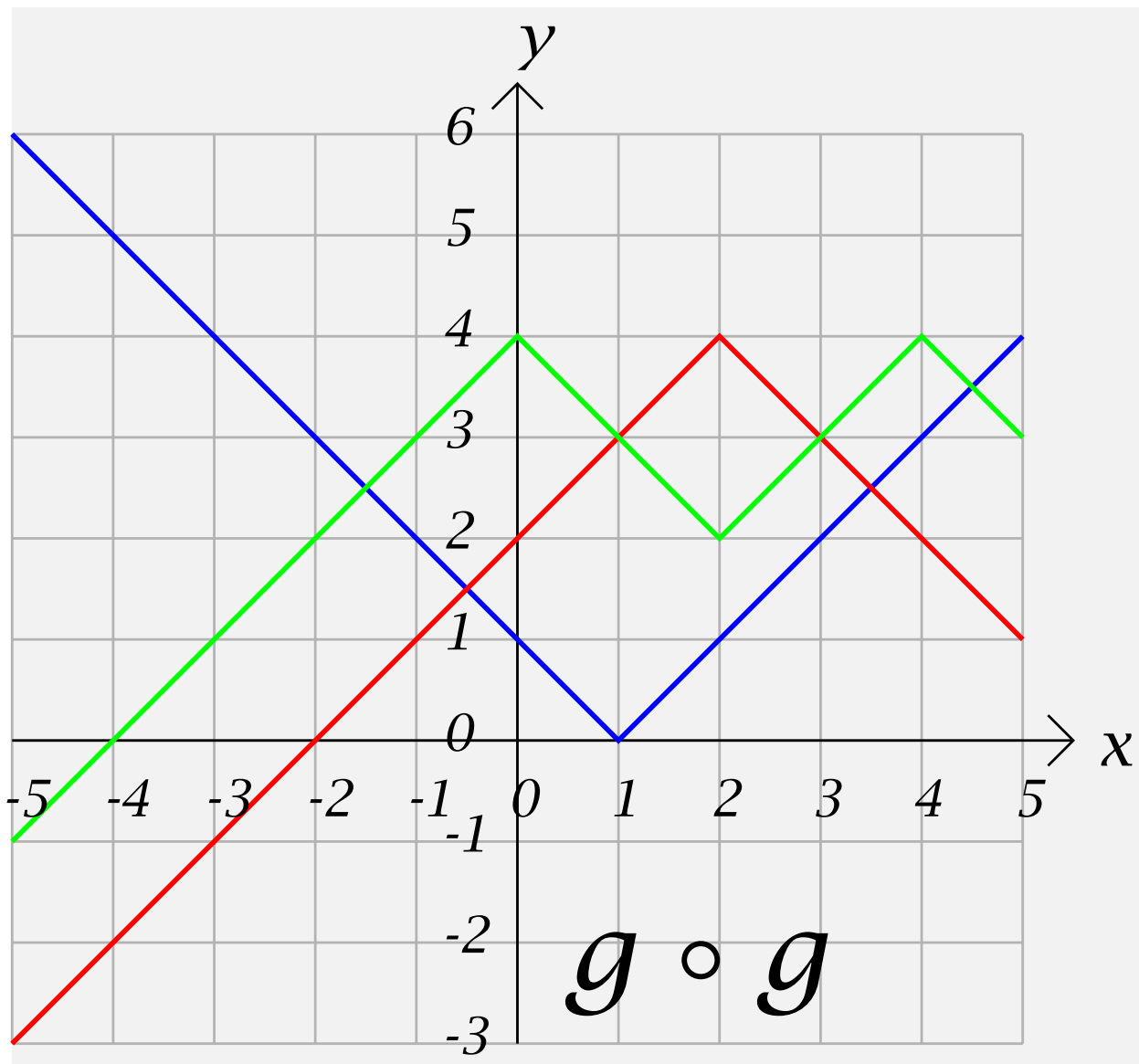
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New Functions From Old Ones-12



New Functions From Old Ones-13



Problems like these have become very common on final exams lately, because they test for **understanding**, not computational ability. Both skills are, of course, very important.

Compositions of three or more functions:

By $f \circ g \circ h(x)$ we mean $f(g(h(x)))$.

Example Let $f(x) = x^2$, $g(x) = 2x + 1$, and $h(x) = \frac{x + 1}{x - 1}$.

Then $(g \circ h)(x) = 2h(x) + 1 = 2\frac{x + 1}{x - 1} + 1$, and

$$(f \circ g \circ h)(x) = ((g \circ h)(x))^2 = \left(2\frac{x + 1}{x - 1} + 1\right)^2$$
