Mathematics 101.3 Final Examination Solutions

PART I

The possible answers to all questions in Part I are the digits in the **ANSWER SET:**

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5 (G) 6 (H) 7 (I) 8 (J) 9

If $\frac{5}{11} - \frac{2}{5}$ is written in its simplest form as $\frac{a}{10b + c}$, where a, b, and c are digits, then

Solution: $\frac{5}{11} - \frac{2}{5} = \frac{5 \cdot 5 - 2 \cdot 11}{11 \cdot 5} = \frac{25 - 22}{55} = \frac{3}{10 \cdot 5 + 5}$

(1) $a = 3$ **(2)** $b = 5$ **(3)** $c = 5$

$2x^2 - 20x + 58$ is to be written in the form $a\left[(x-b)^2 + c^2\right]$ by completing squares. We must have:

Solution: $2x^2 - 20x + 58 = 2\left[x^2 - 10x + 29\right] = 2\left[(x-5)^2 + 2^2\right]$

(4) $a = 2$ **(5)** $b = 5$ **(6)** $c = 2$

$4x^2 + 40x + 96$ is to be written in the form $a(x+b)^2 - c$ by completing squares. We must have:

Solution: $4x^2 + 40x + 96 = 4(x^2 + 10x + 24) = 4((x+5)^2 - 1) = 4(x+5)^2 - 4$

(7) $a = 4$ **(8)** $b = 5$ **(9)** $c = 4$

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(10) If a is the slope of the line through the points $(-4, -12)$ and $(4, 12)$, then a is: 3

Solution: $m = \frac{12 - (-12)}{4 - (-4)} = \frac{24}{8} = 3$

The roots of $2x^2 + x - 6 = 0$ in their simplest form are $-A$ and $\frac{B}{C}$. We must have:

Solution: $x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot (-6)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{49}}{4} = \frac{-1 \pm 7}{4} = -2, \frac{3}{2}$

(11) $A = 2$ **(12)** $B = 3$ **(13)** $C = 2$

The polynomial $p(x) = 10x^4 - 7x^3 - 9x^2 + 7x - 1$ can be factored in the form $(x-a)(x+b)(cx-1)(dx-1)$, where a, b, c, and d are digits, and $c < d$. Their values are:

Solution: Since $p(1) = p(-1) = 0$, $p(x)$ is divisble by $(x-1)(x+1) = x^2 - 1$: $\dfrac{p(x)}{x^2-1} = 10x^2 - 7x + 1$, which factors to $(2x-1)(5x-1)$.

(14) $a = 1$ **(15)** $b = 1$ **(16)** $c = 2$ **(17)** $d = 5$

$\dfrac{3\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} - 2$ can be simplified to the expression $a + b\sqrt{c}$, where a, b, and c are digits. Their values are:

Solution: $\dfrac{3\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - 2 = \dfrac{3\sqrt{3}^2 - 3\sqrt{3}\sqrt{2} - \sqrt{3}\sqrt{2} + \sqrt{2}^2}{\sqrt{3}^2 - \sqrt{2}^2} - 2 = \dfrac{3 \cdot 3 - 4\sqrt{6} + 2}{3 - 2} - 2 = \dfrac{11 - 4\sqrt{6}}{1} - 2 = 11 - 4\sqrt{6} - 2 = 9 - 4\sqrt{6}$

(18) $a = 9$ **(19)** $b = 4$ **(20)** $c = 6$

If we solve the inequality $\left|\dfrac{5-x}{4}\right| \geq 1$, the solution is an interval of the form $(-\infty, a] \cup [b, \infty)$.
The values of a and b are:

Solution: $|5 - x| \geq 4$

(21) $a = 1$ **(22)** $b = 9$

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(23) The x-intercept of the line perpendicular to the line $y = -\dfrac{x}{14}$ which passes through the point $(9, 0)$ is:

Solution: 9

Evaluate the limits:

(24) $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ if $f(x) = x^3$ and $x = \dfrac{2}{\sqrt{3}}$

Solution: $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h} =$

$\lim\limits_{h\to 0} \dfrac{3x^2h+3xh^2+h^3}{h} = \lim\limits_{h\to 0} 3x^2+3xh+h^2 3x^2 = 3\left(\dfrac{2}{\sqrt{3}}\right)^2 = 3\dfrac{4}{3} =$ **4**

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(25) $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ if $f(x) = -\dfrac{7}{x+4}$ and $x = -5$

Solution: $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{-\frac{7}{x+h+4} - \left(-\frac{7}{x+4}\right)}{h} = -7\lim\limits_{h\to 0} \dfrac{\frac{1}{x+h+4} - \frac{1}{x+4}}{h} =$

$-7\lim\limits_{h\to 0} \dfrac{\frac{x+4-(x+h+4)}{(x+4)(x+h+4)}}{h} = -7\lim\limits_{h\to 0} \dfrac{\frac{-h}{(x+4)(x+h+4)}}{h} = 7\lim\limits_{h\to 0} \dfrac{1}{(x+4)(x+h+4)} = 7\dfrac{1}{(x+4)^2} =$
$7\dfrac{1}{(-5+4)^2} =$ **7**

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(26) $\lim\limits_{x\to -3} \dfrac{6x^2+30x-84}{3x-6}$

Solution: $\lim\limits_{x\to -3} \dfrac{6x^2+30x-84}{3x-6} = \dfrac{6(-3)^2+30(-3)-84}{3(-3)-6} = \dfrac{54-90-84}{-9-6} = \dfrac{-120}{-15} =$
8

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(27) $\lim\limits_{x\to -5} \dfrac{-15-8x-x^2}{x+5}$

Solution: $\lim\limits_{x\to -5} \dfrac{-15-8x-x^2}{x+5} = \lim\limits_{x\to -5} \dfrac{-(x+5)(x+3)}{x+5} = \lim\limits_{x\to -5} -(x+3) = -(-5+3) = -(-2) =$ **2**

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(28) The natural domain of the function $f(x) = \sqrt[8]{64x^2-1}$ is of the form $\left(-\infty, -\dfrac{1}{a}\right] \cup \left[\dfrac{1}{a}, \infty\right)$. What is the value of a =?

Solution: $64x^2 - 1 \geq 0$ if $x^2 \geq \dfrac{1}{64} = \dfrac{1}{8^2}$, so $a =$ **8**

(29) The $y-$intercept of the line perpendicular to the line $\frac{x}{6} = y$ which passes through the point $(1, 2)$ is:

Solution: The perpendicular line has slope -6, so its Point-Slope Equation is $y - 2 = -6(x - 1)$. Setting $x = 0$, we get $y = \boxed{8}$

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(30) Let $f(x) = \frac{x^4}{(x+1)^4}$. Find $64f'(1)$

Solution: Since $f(x) = \left(\frac{x}{x+1}\right)^4$, we have $f'(x) = 4\left(\frac{x}{x+1}\right)^3 \frac{(x+1)(1) - x(1)}{(x+1)^2} = 4\left(\frac{x}{x+1}\right)^3 \frac{x+1-x}{(x+1)^2} = 4\frac{x^3}{(x+1)^5}$, so $f'(1) = 4\frac{1^3}{(1+1)^5} = \frac{1}{8}$, and $64f'(1) = 64\frac{1}{8} = \boxed{8}$

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(31) The minimum value of $f(x) = x^2 + 8x + 22$ is:

Solution: $f'(x) = 2x + 8 = 0$ if $x = -4$. $f(-4) = (-4)^2 + 8(-4) + 22 = 16 - 32 + 22 = \boxed{6}$

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(32) Find the positive solution of the equation $2^{x^2-1} = 256$:

Solution: $256 = 2^8$, so we must have $x^2 - 1 = 8$, so $x = \boxed{3}$

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(33) The largest solution of $\ln(x + 3) + \ln(x + 7) = \ln(20x)$ is:

Solution: Since $\ln(x+3)+\ln(x+7) = \ln(x+3)(x+7)$, we must have $(x+3)(x+7) = 20x$, or

$x^2 + 10x + 21 = 20x$, or $x^2 - 10x + 21 = 0$, or $(x - 3)(x - 7) = 0$. The largest solution is thus $\boxed{7}$.

If $\log_2 1024 = 10a + b$ then **(34)** $a = \boxed{1}$ and **(35)** $b = \boxed{0}$

Solution: Since $1024 = 2^{10}$, we have $\log_2 1024 = \log_2 2^{10} = 10$, so $10 = 10a + b$.

If $\log_{25} 125 = \frac{c}{d}$ then **(36)** c = 3 and **(37)** d = 2

Solution: $\log_{25} 125 = \frac{\ln 125}{\ln 25} = \frac{\ln 5^3}{\ln 5^2} = \frac{3 \ln 5}{2 \ln 5} = \frac{3}{2}$

If $\log_{16} 128 = \frac{e}{f}$ then **(38)** e = 7 and **(39)** f = 4

Solution: $\log_{16} 128 = \frac{\ln 128}{\ln 16} = \frac{\ln 2^7}{\ln 2^4} = \frac{7 \ln 2}{4 \ln 2} = \frac{7}{4}$

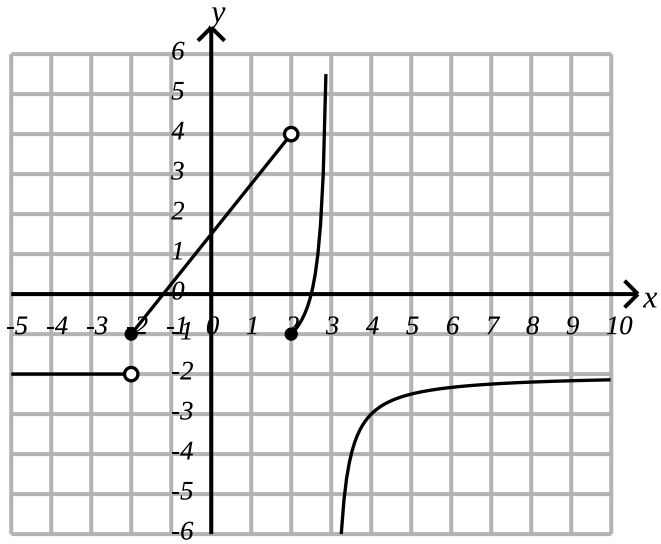
PART II

The possible answers to all questions in Part II are in the **ANSWER SET:**

(A) $-\infty$ (B) -4 (C) -2 (D) -1 (E) 0 (F) 1 (G) 2 (H) 4 (I) 8 (J) ∞

(40) $\lim\limits_{x\to -4^-} \dfrac{6x-1}{x+4}$ =? J

(41) $\lim\limits_{x\to -4^+} \dfrac{6x-1}{x+4}$ =? A



Part of the graph of $y = f(x) = \begin{cases} -2 & \text{if } x < -2 \\ \dfrac{5x+6}{4}x & \text{if } -2 \le x < 2 \\ \dfrac{1}{3-x} - 2 & \text{if } 2 \le x \text{ and } x \neq 3 \end{cases}$ is shown above.

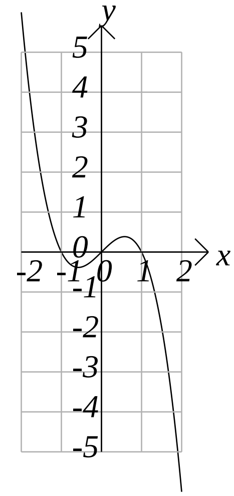
Find:

(42) $\lim\limits_{x\to -\infty} f(x)$ C **(43)** $\lim\limits_{x\to -2^-} f(x)$ C **(44)** $\lim\limits_{x\to -2^+} f(x)$ D **(45)** $\lim\limits_{x\to 2^-} f(x)$ H

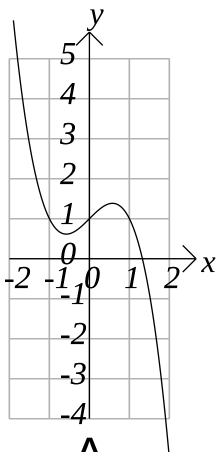
(46) $\lim\limits_{x\to 2^+} f(x)$ D **(47)** $\lim\limits_{x\to 3^-} f(x)$ J **(48)** $\lim\limits_{x\to 3^+} f(x)$ A **(49)** $\lim\limits_{x\to \infty} f(x)$ C

PART III

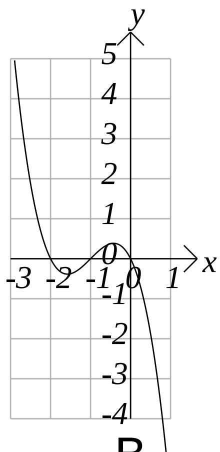
The graph of $y = f(x) = x - x^3$ is shown to the right. Parts of the graphs of



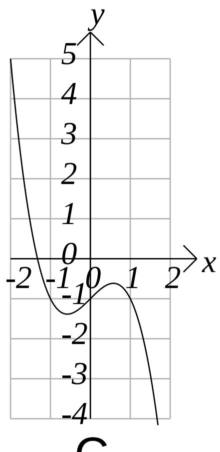
(50) $y = f(2x)$, G
(51) $y = 2f(x)$, F
(52) $y = f(x)/2$, H
(53) $y = f(x+1)$, B
(54) $y = f(x) - 1$, C
(55) $y = f(x-1)$, D
(56) $y = -f(x)$, E
(57) $y = f(x/2)$, and I
(58) $y = -f(x) + 1$, J are shown below.
Match them.



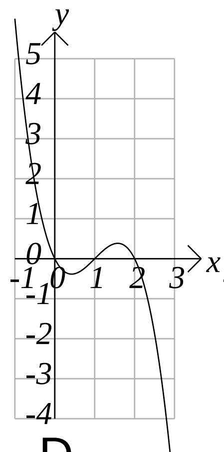
A



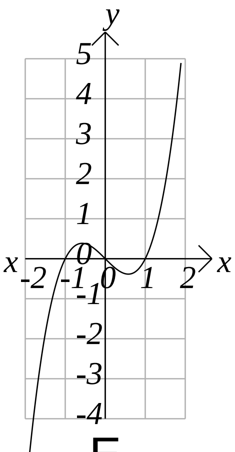
B



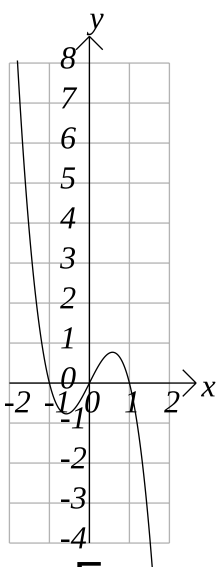
C



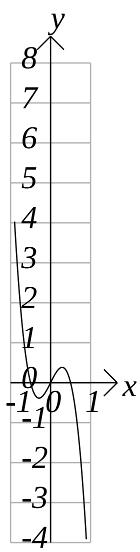
D



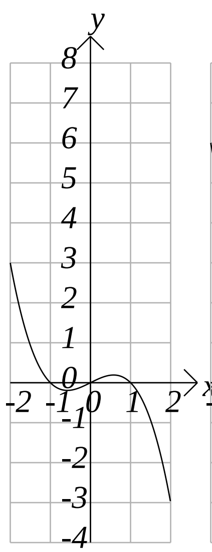
E



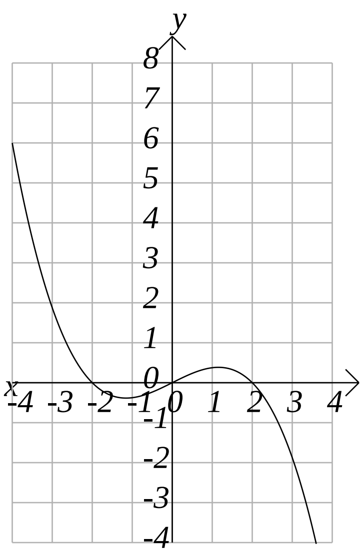
F



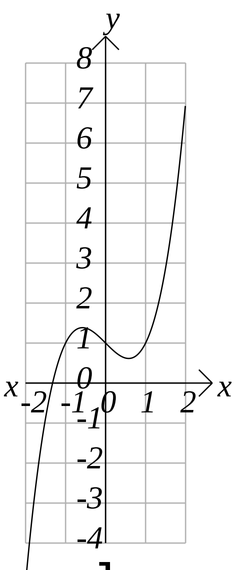
G



H



I



J

PART IV

The graphs of the functions in questions 59 to 65 are shown below. Match them.

(59) $f(x) = \frac{1}{10}\left(2x^3 - 6x^2 + 3\right)$ D

(60) $f(x) = \frac{1}{10}\left(x^4 - 4x^2\right)$ E

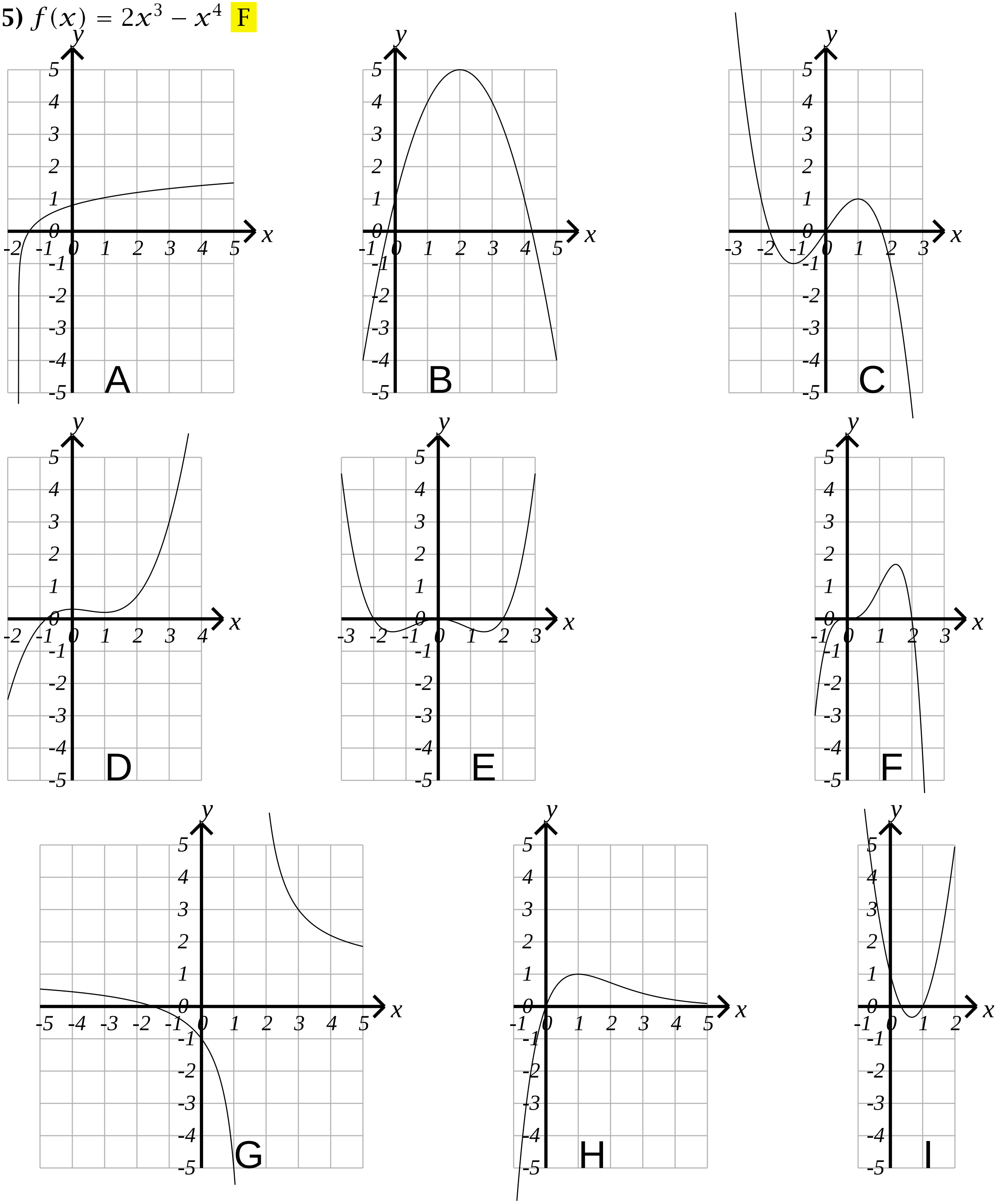
(61) $f(x) = \frac{1}{2}\left(3x - x^3\right)$ C

(62) $f(x) = \frac{2x+3}{2x-3}$ G

(63) $f(x) = 4x - x^2 + 1$ B

(64) $f(x) = 3x^2 - 4x + 1$ I

(65) $f(x) = 2x^3 - x^4$ F



PART V

The graphs of the functions in questions 66 to 73 are shown below. Match them.

(66) $f(x) = 5e^{-\frac{1}{x^2}}$ H

(67) $f(x) = xe^{x}$ A

(68) $f(x) = 5x^2e^{-2x}$ C

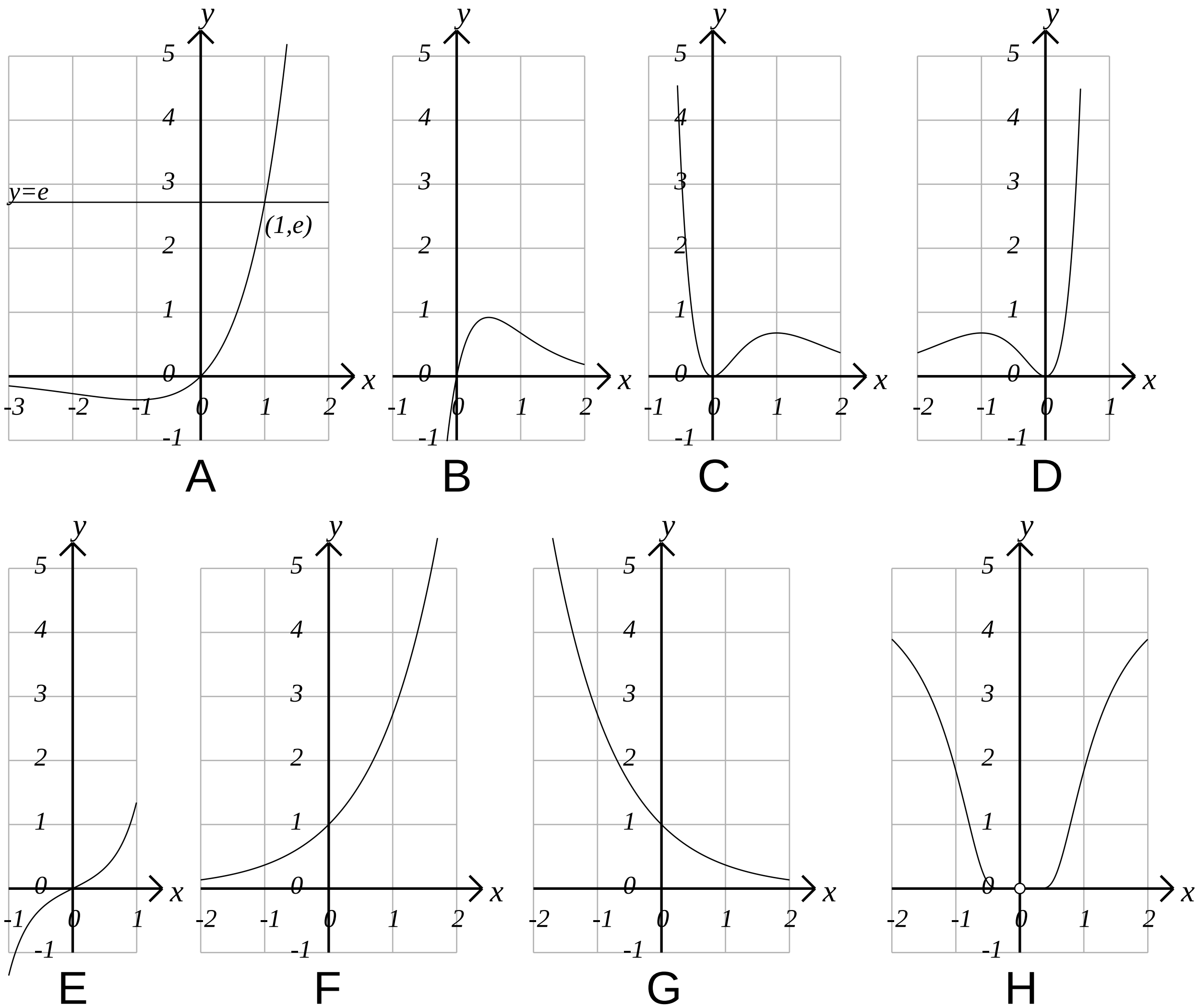
(69) $f(x) = 5x^2e^{2x}$ D

(70) $f(x) = 5xe^{-2x}$ B

(71) $f(x) = 0.5 * xe^{x^2}$ E

(72) $f(x) = e^{x}$ F

(73) $f(x) = e^{-x}$ G



PART VI

The graphs of the functions in questions 74 to 80 are shown below. Match them.

(74) $f(x) = \ln x^x$ B

(75) $f(x) = \ln x$ A

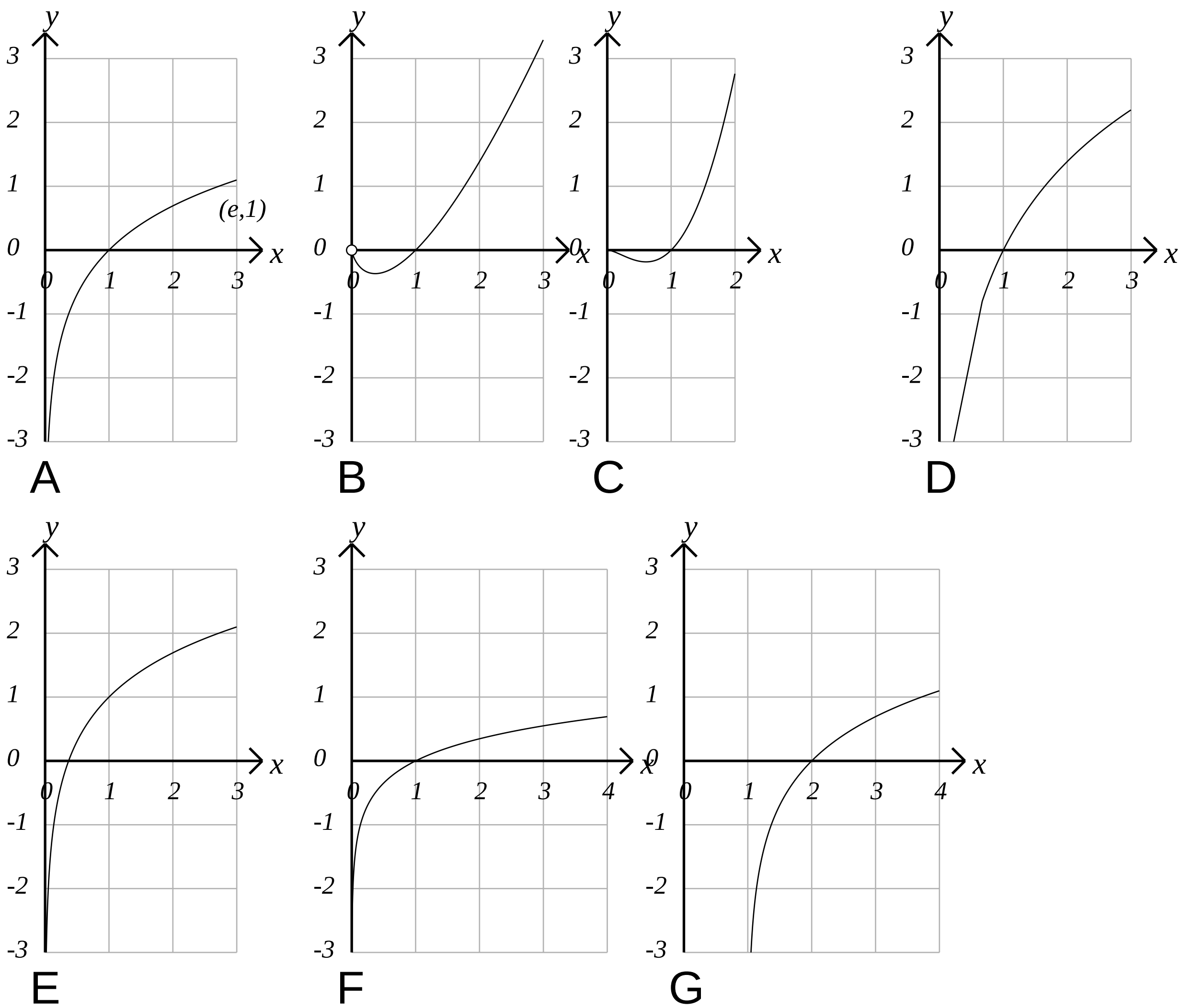
(76) $f(x) = \ln \sqrt{x}$ F

(77) $f(x) = x \ln x$ B

(78) $f(x) = \ln x^2$ D

(79) $f(x) = x^2 \ln x$ C

(80) $f(x) = \ln ex$ E



PART VII

Each question in this section is worth 5 marks.

The possible answers to all questions in Part VII are the digits in the **ANSWER SET:**

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5 (G) 6 (H) 7 (I) 8 (J) 9

(81) Find the absolute minimum value of the function $f(x) = 9 + (\ln 3x)^2$.

Solution: $f'(x) = 2(\ln 3x)\dfrac{3}{3x} = 0$ if $\ln 3x = 0$, or $3x = 1$. The minimum is $\boxed{9}$.

(82) Two positive numbers have their product equal to 16. What is the minimum possible value of their sum?

Solution: Let the numbers be x and y, so that $xy = 16$ and $y = 16x^{-1}$. The sum is $f(x) = x + 16x^{-1}$. Then $f'(x) = 1 + 16(-1)x^{-2} = 0$ if $x = 4$. The minimum sum is thus $4 + 4 = \boxed{8}$.

(83) A spoonful of hot (90°C) water is placed in a room whose temperature is 10°C. In one minute its temperature drops to 50°C. How many more minutes will it take for its temperature to reach 15°C?

Hint: The formula $T = T_\infty + (T_0 - T_\infty)e^{kt}$ applies.

Solution: We have $T = 10 + (90-10)e^{kt} = 10 + 80e^{kt}$ and $T(1) = 50 = 10 + 80e^{k\cdot 1}$, so

$50 - 10 = 40 = 80e^k$, and $e^k = \dfrac{1}{2}$. Taking logs, we get $k = \ln\dfrac{1}{2} = -\ln 2$. Thus

$T(t) = 10 + 80e^{(-\ln 2)t}$ which equals 15 if $15 = 10 + 80e^{(-\ln 2)t}$ or $5 = 80e^{(-\ln 2)t}$, or

$\dfrac{1}{16} = e^{(-\ln 2)t}$. Taking logs again, we get

$\ln\left(\dfrac{1}{16}\right) = (-\ln 2)t$, of $-4\ln 2 = (-\ln 2)t$, so $t = 4$. Therefore it takes $\boxed{3}$ more minutes.

(84) What is the area between the graph of $y = 4x^3 + 3x^2 + 2$ and the x-axis from $x = 0$ to $x = 1$? **Solution:** $\displaystyle\int_0^1 (4x^3 + 3x^2 + 2)dx = x^4 + x^3 + 2x\Big|_0^1 = (1^4 + 1^3 + 2(1)) - (0^4 + 0^3 + 2(0)) = 1 + 1 + 2 - 0 = \boxed{4}$