# Techniques for Finding Derivatives

## Derivative Rules:



(1) If $c$ is a constant, then $\frac{d}{dx}[c] = 0.$

(2) If $a$ is any real number, $\frac{d}{dx}[x^a] = ax^{a-1}.$

(3) If $f$ is differentiable at $x$, $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].$

If $f$ and $g$ are differentiable at $x$, then

(4) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)].$

(5) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$

These rules are sufficient for the differentiation of all polynomials.

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**Example 1:** $\frac{d}{dx}[8x^{10}] = 8\frac{d}{dx}[x^{10}] = 8\left(10x^{10-1}\right) =$ $80x^9$

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**Example 2:** $\frac{d}{dx}[x^5 + x^{-1}] = \frac{d}{dx}[x^5] + \frac{d}{dx}[x^{-1}] =$

$5x^{5-1} + (-1)x^{-1-1} =$ $5x^4 - x^{-2}$

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Example 3: $\frac{d}{dx}[5x^4 - x^{-2}] = \frac{d}{dx}[5x^4] - \frac{d}{dx}[x^{-2}] =$

$5\frac{d}{dx}[x^4] - (-2)x^{-2-1} = 5(4)x^{4-1} + 2x^{-3} =$ $20x^3 + 2x^{-3}$

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# The Product & Quotient Rules

## The Product Rule

The derivative of the product of two non-constant functions is never equal to the product of their derivatives.

We have $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

or, in function notation

$$(fg)' = fg' + f'g$$

or

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$$

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Example 4: Let $f(x) = x$, and $g(x) = x$, so that $f(x)g(x) = x \cdot x = x^2$. Then the Product Rule gives us

$(x \cdot x)' = x(x)' + (x)'x = x(1) + (1)x = 2x$, which agrees with $(x^2)' = 2x$.

Notice that $f'(x)g'(x) = 1(1) = 1$, which is quite different, and incorrect.

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**Example 5:** Find the derivative of $(2x^2+3)(4x+5)$.

Let $f(x)=2x^2+3$, and $g(x)=4x+5$.

Then $f'(x)=4x$ and $g'(x)=4$, so $f(x)g(x)=(2x^2+3)(4x+5)$, and

$(f(x)g(x))'=f(x)g'(x)+f'(x)g(x)=(2x^2+3)(4)+(4x)(4x+5)=(8x^2+12)+(16x^2+20x)=$ $24x^2+20x+12$

We often write this calculation in a shorter form:

$(2x^2+3)(4x+5)'+(2x^2+3)'(4x+5)=(2x^2+3)(4)+(4x)(4x+5)=$ $24x^2+20x+12$

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**Example 6:** Find the derivative of $(x^2+3x)(x^2+2)$.

Let $f(x)=x^2+3x$, and $g(x)=x^2+2$.

Then $f'(x)=2x+3$ and $g'(x)=2x$, so

$f(x)g(x)=(x^2+3x)(x^2+2)$, and

$(f(x)g(x))'=f(x)g'(x)+f'(x)g(x)=$
$(x^2+3x)(2x)+(2x+3)(x^2+2)=$
$2x^3+6x^2+2x^3+4x+3x^2+6=$
$4x^3+9x^2+4x+6$

We write this in a shorter form:

$(x^2+3x)(x^2+2)'+(x^2+3x)'(x^2+2)=$

$(x^2+3x)(2x)+(2x+3)(x^2+2)=$

$2x^3 + 6x^2 + 2x^3 + 4x + 3x^2 + 6 =$
$4x^3 + 9x^2 + 4x + 6$

## The Quotient Rule

The derivative of the quotient of two non-constant functions requires an even more complicated formula:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

or, in function notation,

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

or

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

**Example 7:** Let $f(x) = x + 1$, and $g(x) = x - 1$, so that $\frac{f(x)}{g(x)} = \frac{x+1}{x-1}$. Then the Quotient Rule gives us:

$$\left(\frac{x+1}{x-1}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} =$$

$$\frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} =$$

$$\frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} =$$

$$\frac{(x-1)-(x+1)}{(x-1)^2} =$$

$$\frac{x-1-x-1}{(x-1)^2} = \boxed{\frac{-2}{(x-1)^2}}.$$

Notice that $\dfrac{f'(x)}{g'(x)} = \dfrac{1}{1} = 1$, which is quite different, and incorrect.

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**Example 8:** Differentiate $\dfrac{x^2-4x+2}{x+3}$

**Solution:** $\left(\dfrac{x^2-4x+2}{x+3}\right)' =$

$$\frac{(x+3)(x^2-4x+2)' - (x^2-4x+2)(x+3)'}{(x+3)^2} =$$

$$\frac{(x+3)(2x-4) - (x^2-4x+2)(1)}{(x+3)^2} =$$

$$\frac{2x^2+2x-12-x^2+4x-2}{(x+3)^2} = \boxed{\frac{x^2+6x-14}{(x+3)^2}}$$

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**Example 9:** Differentiate $\dfrac{\sqrt{t}}{2t+3}$

**Solution:** $\left(\dfrac{\sqrt{t}}{2t+3}\right)' = \left(\dfrac{t^{\frac{1}{2}}}{2t+3}\right)'$

$$\frac{(2t+3)\left(t^{\frac{1}{2}}\right)' - \left(t^{\frac{1}{2}}\right)(2t+3)'}{(2t+3)^2} =$$

$$\frac{(2t+3)\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}(2)}{(2t+3)^2} = \frac{(2t+3)\frac{1}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} =$$

$$\frac{\frac{t}{\sqrt{t}} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\sqrt{t} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2} =$$

$$\frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2}\left(\frac{2\sqrt{t}}{2\sqrt{t}}\right) = \quad \frac{3-2t}{2\sqrt{t}(2t+3)^2}$$