

## The Chain Rule

Recall that the **composite function** or **composition** of two functions is the function obtained by applying them one after the other.

For example, if  $f(x) = \frac{1}{x}$  and  $g(x) = x^3 + 2$ , then

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{x^3 + 2}$$

$$\text{and } g(f(x)) = (f(x))^3 + 2 = \left(\frac{1}{x}\right)^3 + 2 = \frac{1}{x^3} + 2$$

It is important for the student to be able to recognize that a function is the composition of two or more functions, so that the upcoming “Chain Rule” for differentiating composites may be used.

For example, if  $h(x) = \frac{1}{x^3 + 2}$ , we see that we can let  $u = g(x) = x^3 + 2$  to get  $h(x) = \frac{1}{u} = \frac{1}{g(x)}$ .

Then we let  $f(x) = \frac{1}{x}$ , so that  $h(x) = f(u) = f(g(x))$

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The derivative of the composition of two non-constant functions is equal to the product of their derivatives, evaluated appropriately.

We have, in function notation,

$$(g(h(x)))' = g'(h(x))h'(x)$$

or, in Leibnitz notation, if  $y = f(g(x))$  and  $u = g(x)$ ,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We may also write

$$\frac{d}{dx} [f(u)] = f'(u) \frac{du}{dx}$$

**Example 1:** Using  $g(x) = \frac{1}{x} = x^{-1}$  and  $h(x) = x^3 + 2$ ,

we have  $g'(x) = (-1)x^{-2}$  and  $h'(x) = 3x^2$ ,

$g'(h(x)) = (-1)(h(x))^{-2}$ , so we get

$$\left(\frac{1}{x^3 + 2}\right)' = g'(h(x))h'(x) = (-1)(h(x))^{-2}(3x^2) =$$

$$(-1)(x^3 + 2)^{-2}(3x^2) = \frac{-3x^2}{(x^3 + 2)^2}$$

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**Example 2:** Let  $g(x) = x^3$ , and  $h(x) = x^2$ , so that

$$g(h(x)) = (h(x))^3 = (x^2)^3 = x^6.$$

Then  $g'(x) = 3x^2$ , so  $g'(h(x)) = 3(h(x))^2$ , and  $h'(x) = 2x$ ,

so the Chain Rule gives us

$$(g(h(x)))' = g'(h(x))h'(x) = (3(h(x))^2)(2x) =$$

$$(3(x^2)^2)(2x) = (3x^4)(2x) = 6x^5, \text{ as expected.}$$

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**Example 3:** Let  $g(x) = x^3 + 3$ , and  $h(x) = x^2 + 2$ , so that

$$g(h(x)) = (h(x))^3 + 3 = (x^2 + 2)^3 + 3.$$

Then  $g'(x) = 3x^2$ , so  $g'(h(x)) = 3(h(x))^2$ , and  $h'(x) = 2x$ ,

so the Chain Rule gives us

$$(g(h(x)))' = g'(h(x))h'(x) = (3(h(x))^2)(2x) = (3(x^2 + 2)^2)(2x) =$$

$$6x(x^2 + 2)^2$$

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**Example 4:** Find  $f'(x)$  if  $f(x) = \sqrt[3]{x^4 + x^2 + 1}$ .

We let  $g(x) = x^{\frac{1}{3}}$  and  $u = x^4 + x^2 + 1$  so that  $f(x) = g(u)$ .

$$\text{Then } g'(x) = \frac{1}{3}x^{-\frac{2}{3}}, \quad g'(u) = \frac{1}{3}u^{-\frac{2}{3}}, \quad \text{and } \frac{du}{dx} = 4x^3 + 2x,$$

$$\text{so we have } f'(x) = g'(u) \frac{du}{dx} = \frac{1}{3}(u)^{-\frac{2}{3}}(4x^3 + 2x) =$$

$$\frac{2x(2x^2 + 1)}{3(x^4 + x^2 + 1)^{\frac{2}{3}}}$$

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## The Power Rule

$$((g(x))^n)' = n(g(x))^{n-1}g'(x)$$

**Example 4a:** Find  $f'(x)$  if  $f(x) = \sqrt[3]{x^4 + x^2 + 1}$ .

We write  $f(x) = (g(x))^{\frac{1}{3}}$  where  $g(x) = x^4 + x^2 + 1$ .

$$\text{Then } f'(x) = \frac{1}{3}(g(x))^{-\frac{2}{3}}g'(x) = \frac{1}{3}(x^4 + x^2 + 1)^{-\frac{2}{3}}(4x^3 + 2x) =$$

$$\frac{2x(2x^2 + 1)}{3(x^4 + x^2 + 1)^{\frac{2}{3}}}$$

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**Example 5:** Find  $f'(x)$  if  $f(x) = \left(\frac{4x-3}{2x+1}\right)^8$ .

$$\text{We have } f'(x) = 8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{4x-3}{2x+1}\right)' =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{(2x+1)(4x-3)' - (4x-3)(2x+1)'}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{(2x+1)4 - (4x-3)2}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{8x+4-8x+6}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{10}{(2x+1)^2}\right) = 80\frac{(4x-3)^7}{(2x+1)^9}$$

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**Example 6:** Find  $f'(x)$  if  $f(x) = \frac{(4x-3)^8}{(2x+1)^5}$ .

First we use the Quotient Rule:

$$f'(x) = \frac{[(2x+1)^5][(4x-3)^8]' - (4x-3)^8[(2x+1)^5]'}{[(2x+1)^5]^2}$$

and then the Power Rule:

$$f'(x) =$$

$$\frac{[(2x + 1)^5] 8(4x - 3)^{8-1}(4x - 3)' - (4x - 3)^8 5(2x + 1)^{5-1}(2x + 1)'}{(2x + 1)^{5 \times 2}} =$$

$$\frac{(2x + 1)^5 8(4x - 3)^7(4) - (4x - 3)^8 5(2x + 1)^4(2)}{(2x + 1)^{10}} =$$

$$\frac{32(2x + 1)^5(4x - 3)^7 - 10(4x - 3)^8(2x + 1)^4}{(2x + 1)^{10}} =$$

$$\frac{32(2x + 1)(4x - 3)^7 - 10(4x - 3)^8}{(2x + 1)^6} = (4x - 3)^7 \frac{32(2x + 1) - 10(4x - 3)}{(2x + 1)^6} =$$

$$\frac{(4x - 3)^7}{(2x + 1)^6} [64x + 32 - 40x + 30] = \frac{(4x - 3)^7}{(2x + 1)^6} [24x + 62] =$$

$$2 \frac{(4x - 3)^7}{(2x + 1)^6} [12x + 31]$$

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