Derivatives

Definition: If the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists, it is called the derivative of f at a,

and is denoted by $f'(a)$,

pronounced "f prime of a."

Example 1: $f(x) = x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} =$$

$$\lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0} \frac{(2a+h)h}{h} =$$

$$\lim_{h\to 0} 2a + h = 2a + 0 = 2a$$

Note: This tells us that the slope of the tangent line to the parabola $y = x^2$ at any point (x, x^2) is always twice the x-coordinate.

Example 2: $f(x) = x^3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$\lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} =$$

$$\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}\frac{(3x^2+3xh+h^2)h}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2+3x(0)+0^2 = \boxed{3x^2}$$

Example 3: Find the equation of the tangent line to $y = f(x) = 5x^2 - 3x$, at $(2, 14)$,

$$f(a) = f(2) = 5(2)^2 - 3(2) = 5(4) - 6 = 20 - 6 = 14$$

$$m_{\mathbf{PQ}} = \frac{f(a+h)-f(a)}{h} = \frac{f(2+h)-f(2)}{h} = \frac{5(2+h)^2-3(2+h)-14}{h} =$$

$$\frac{5(4+4h+h^2)-6-3h-14}{h} = \frac{20+20h+5h^2-6-3h-14}{h} =$$

$$\frac{20h+5h^2-3h}{h} = \frac{17h+5h^2}{h} = 17+5h$$

Taking limits,

$$m = \lim_{h\to 0} m_{\mathbf{PQ}} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} 17+5h = 17+5(0) = 17$$

Therefore the equation of the tangent line is $\boxed{y - 14 = 17(x-2)}$

Example 4: $f(x) = \dfrac{1}{x}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} =$$

$$\lim_{h\to 0}\frac{\frac{1}{x+h}\left(\frac{x}{x}\right)-\frac{1}{x}\left(\frac{x+h}{x+h}\right)}{h} =$$

$$\lim_{h\to 0} \frac{\dfrac{x}{x(x+h)} - \dfrac{x+h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\dfrac{x-(x+h)}{x(x+h)}}{h} =$$

$$\lim_{h\to 0} \frac{\dfrac{-h}{x(x+h)}}{\dfrac{h}{1}} = \lim_{h\to 0} \left(\frac{-h}{x(x+h)}\right)\left(\frac{1}{h}\right) = \lim_{h\to 0} \frac{-1}{x(x+h)} =$$

$$\frac{-1}{x(x+0)} = \frac{-1}{x^2} \quad \text{if } x \neq 0.$$

Remember: $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \left(\dfrac{a}{b}\right)\left(\dfrac{d}{c}\right)$

The Derivative as a Function

The slope defines a new function $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$, called the derivative of f. Since the limit doesn't necessarily exist, the derivative may have a smaller domain than f.

Definition: We define the derivative of f to be the function f' defined by the rule

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

whenever the limit exists.

Example 5: If $f(x) = x^4 - x^2$, find a formula for $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^4 - (x+h)^2] - [x^4 - x^2]}{h} =$$

$$\lim_{h\to 0} \frac{[x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^2 - 2xh - h^2] - x^4 + x^2}{h} =$$

$$\lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2xh - h^2}{h} =$$

$$\lim_{h\to 0} \frac{(4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h)h}{h} =$$

$$\lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h = \quad 4x^3 - 2x$$

Other Notations

The derivative of $y = f(x)$ is written in many different ways in the many branches of Science that use derivatives:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = D_x y = D_x(f(x))$$

Some Useful Theory

Definition: If the derivative of f exists at x, we say that f is differentiable at x. f is differentiable on an open interval (a, b) if it is differentiable at every $x \in (a, b)$.

We have already observed that a limit of a quotient where the denominator approaches zero cannot exist unless the numerator also approaches zero.

In the case of the limit of a quotient that we use to define the derivative function, the denominator h always approaches zero, so the numerator must also approach zero if the derivative is to exist.

In other words, if the derivative exists, we have $\lim_{h\to 0} f(x+h) - f(x) = 0$, or

$\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x) = f(x)$, so we have the

Theorem: If f is differentiable at x, it is also continuous at x.

Note that the converse is not always true: try $f(x) = |x|$ at $x = 0$.

What Goes On When Differentiability Fails?

First, there might be a discontinuity of the function. If the function is continous, then the left- or right-hand limits $\lim_{h\to 0-} \frac{f(x+h)-f(x)}{h}$ and $\lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$ may be unequal, be infinite, or just plain fail to exist in any sense at all.