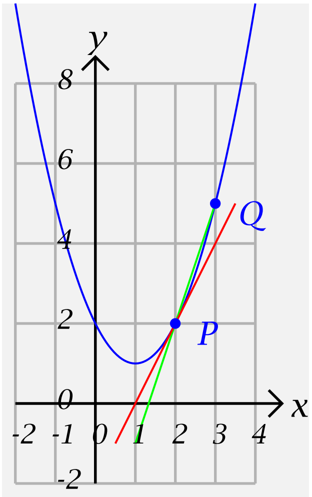
Tangent Lines

In geometry, the tangent line to a circle with centre **O** at a point **A** on the circle is defined to be the perpendicular line at **A** to the line **OA.**
The tangent lines have the special property that they intersect the circle in exactly one point. Any curve that can be obtained from a circle by projection (casting shadows) has tangent lines that are the projections (shadows) of the tangent lines to the circle. These curves are ellipses, parabolas, and hyperbolas, and their tangents have the same intersection property.

If we wish to define tangent lines to other curves, we cannot expect the single intersection property to still hold true. If we are given the graph of a function with equation $y = f(x)$, and wish to define the tangent to the graph at a point **P**$= (a, f(a))$ on the graph, then all we need is to define its slope m, and we can use the point-slope form $y - f(a) = m(x - a)$

Slope of a Tangent Line



If $P = (a, f(a))$ is a point on the graph of a continuous function f, the secant line through P and another point $Q = (b, f(b))$ has slope

$$m_{PQ} = \frac{f(b) - f(a)}{b - a}.$$

If $\lim\limits_{Q \to P} m_{PQ} = \lim\limits_{b \to a} \frac{f(b) - f(a)}{b - a}$ exists and equals m,

we define the tangent line to $y = f(x)$ at P to be the line through P with slope m.

The Point-Slope Form equation of this tangent line is

$$y - f(a) = m(x - a)$$

We can rewrite the limit calculation, using $h = b - a$ and $b = a + h$, in the form

$$\lim_{Q \to P} m_{PQ} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

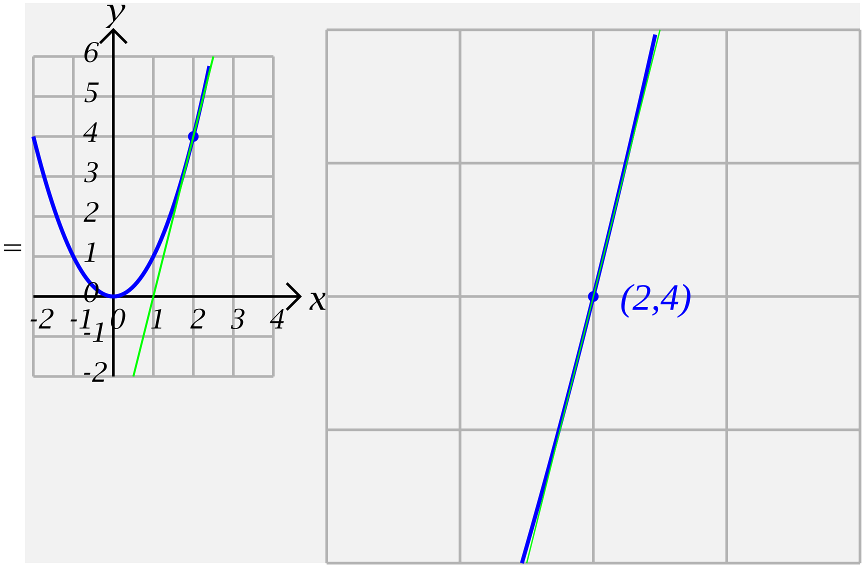
Example 1: $f(x) = x^2$, $P = (2, 4)$.

$$m_{PQ} = \frac{f(b) - f(a)}{b - a} = \frac{f(b) - f(2)}{b - 2} = \frac{b^2 - 2^2}{b - 2} = \frac{(b-2)(b+2)}{b-2} = b + 2,$$

so $m = \lim_{Q \to P} m_{PQ} = \lim_{b \to 2}(b + 2) = \lim_{b \to 2} b + \lim_{b \to 2} 2 = 2 + 2 = 4$.

Alternative limit formulation:

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{2^2 + 2(2)h + h^2 - 2^2}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0}(4 + h) = \lim_{h \to 0} 4 + \lim_{h \to 0} h = 4 + 0 = 4$$



The equation of the tangent line is $y - 4 = 4(x - 2)$

Example 2: $f(x) = x^2$, $P = (a, a^2)$.

$$m_{PQ} = \frac{f(b) - f(a)}{b - a} = \frac{b^2 - a^2}{b - a} = \frac{(b-a)(b+a)}{b-a} = b + a, \text{ so}$$

$m = \lim_{Q \to P} m_{PQ} = \lim_{b \to a} b + a$ exists (and equals $\lim_{b \to a} b + \lim_{b \to a} a = a + a = 2a$ for any number a.)

If we take a to be 1, then $m = \lim_{Q \to P} m_{PQ} = \lim_{b \to a}(b + 1) = 1 + 1 = 2$,

so the Point-Slope Form equation of the tangent line to the curve $y = x^2$ at the point $(1, 1)$ is $y - 1 = 2(x - 1)$

Since the slope of the tangent line at (a, a^2) is $2a$, its equation in Point-Slope Form is $y - a^2 = 2a(x - a)$

Using the alternative limit formulation, we have, for arbitrary a:

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2-a^2}{h} =$$

$$\lim_{h\to 0} \frac{a^2+2ah+h^2-a^2}{h} = \lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0}(2a+h) = \lim_{h\to 0} 2a + \lim_{h\to 0} h = 2a+0 = 2a$$

Example 3: $f(x) = \frac{1}{x}, P = \left(2, \frac{1}{2}\right).$

$$m_{PQ} = \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(2)}{b-2} = \frac{\frac{1}{b}-\frac{1}{2}}{b-2} =$$

$$\frac{\left(\frac{1}{b}\right)\left(\frac{2}{2}\right)-\left(\frac{b}{b}\right)\left(\frac{1}{2}\right)}{b-2} = \frac{\left(\frac{2}{2b}\right)-\left(\frac{b}{2b}\right)}{b-2} = \frac{\frac{2-b}{2b}}{b-2} = -\frac{b-2}{2b}\frac{1}{b-2} = -\frac{1}{2b},$$

so $m = \lim_{Q\to P} m_{PQ} = \lim_{b\to 2} -\frac{1}{2b} = -\frac{1}{2(2)} = -\frac{1}{4}$ exists,

and thus the equation of the tangent line is

$y - \frac{1}{2} = -\frac{1}{4}(x-2)$

Using the alternative limit formulation, we have

$$m = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} =$$

$$\lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} =$$

$$\lim_{h\to 0} \frac{\frac{1}{2+h}\frac{2}{2}-\frac{2+h}{2+h}\frac{1}{2}}{h} =$$

$$\lim_{h\to 0} \frac{\dfrac{2}{(2+h)2} - \dfrac{2+h}{(2+h)2}}{h} =$$

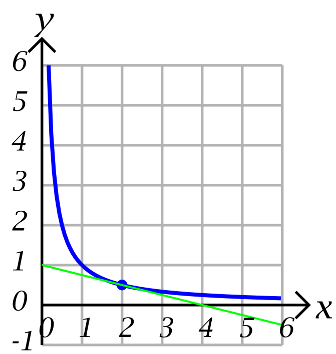
$$\lim_{h\to 0} \frac{\dfrac{2-(2+h)}{(2+h)2}}{h} =$$

$$\lim_{h\to 0} \frac{\dfrac{-h}{(2+h)2}}{h} =$$

$$\lim_{h\to 0} \frac{-h}{(2+h)2}\frac{1}{h} =$$

$$\lim_{h\to 0} \frac{-1}{(2+h)2} =$$

$$\frac{-1}{(2+0)2} = -\frac{1}{4}$$



Example 4: $f(x) = \sqrt{x}$, $P = (4, 2)$.

$$m_{PQ} = \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(4)}{b-4} = \frac{\sqrt{b}-\sqrt{4}}{b-4} = \frac{\sqrt{b}-2}{b-4} =$$

$$\frac{\sqrt{b}-2}{b-4}\left(\frac{\sqrt{b}+2}{\sqrt{b}+2}\right) = \frac{(\sqrt{b})^2-2^2}{(b-4)(\sqrt{b}+2)} = \frac{b-4}{(b-4)(\sqrt{b}+2)} = \frac{1}{\sqrt{b}+2}, \text{ so}$$

$$\lim_{Q\to P} m_{PQ} = \lim_{b\to 4} \frac{1}{\sqrt{b}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4} \text{ exists,}$$

so the equation of the tangent line is

$$y - 2 = \tfrac{1}{4}(x-4)$$

Using the alternative limit formulation, we have

$$m = \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} =$$

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\left(\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right) =$$

$$\lim_{h\to 0}\frac{(\sqrt{4+h})^2-2^2}{h(\sqrt{4+h}+2)} =$$

$$\lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)} =$$

$$\lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)} =$$

$$\lim_{h\to 0}\frac{h}{(\sqrt{4+h}+2)}\frac{1}{h} =$$

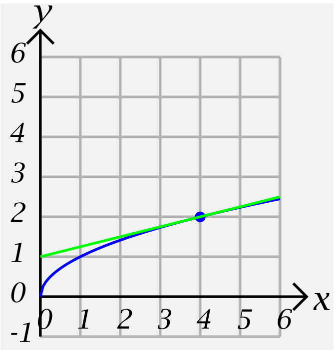
$$\lim_{h\to 0}\frac{1}{(\sqrt{4+h}+2)} =$$

$$\frac{1}{(\sqrt{4+0}+2)} =$$

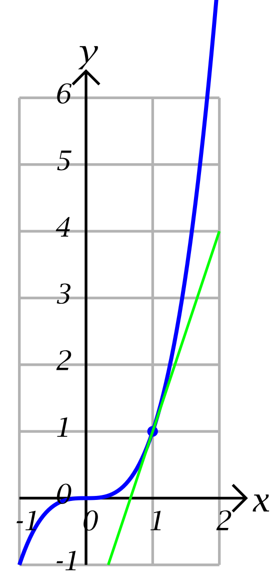
$$\frac{1}{(\sqrt{4+0}+2)} =$$

$$\frac{1}{(2+2)} =$$

$$\frac{1}{4}$$



Example 5: $f(x) = x^3$, $P = (1,1)$.



$$m_{PQ} = \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(1)}{b-1} = \frac{b^3-1^3}{b-1} =$$

$$\frac{(b-1)(b^2+b+1)}{b-1} = b^2+b+1, \text{ so}$$

$$\lim_{Q\to P} m_{PQ} = \lim_{b\to 1} = b^2+b+1 = 3 \text{ exists,}$$

so the equation of the tangent line is

$y - 1 = 3(x-1)$

Using the alternative limit formulation, we have

$$m = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{(1+h)^3-1^3}{h} =$$

$$\lim_{h\to 0}\frac{1^3+3(1)^2h+3(1)h^2+h^3-1^3}{h} = \lim_{h\to 0}\frac{3h+3h^2+h^3}{h} =$$

$$\lim_{h\to 0}\frac{h(3+3h+h^2)}{h} = \lim_{h\to 0}(3+3h+h^2) = (3+3(0)+0^2) = 3$$

Velocity

The Average Velocity of a moving object is defined to be the Distance Travelled divided by the Time Elapsed :

Average Velocity $=$ $\dfrac{\textbf{Distance Travelled}}{\textbf{Time Elapsed}}$

The Average Speed is the absolute value of the average velocity. So, if we drive 120 km in 90 minutes, our average speed and velocity are both equal to:

$$\frac{120\text{km}}{90\text{min}} = \frac{4}{3}\frac{\text{km}}{\text{min}} = \frac{4}{3}\frac{\text{km}}{\text{min}}\frac{60\text{min}}{1\text{hour}} = 80\frac{\text{km}}{\text{hour}}$$

If the moving object is assumed to be moving on a straight line, and its distance from a fixed reference point at time t is given by a function $s(t)$, then the average velocity from time t_1 to time t_2 will be

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Example: $s(t) = 2t^2 - 6t + 3$

Time Interval = $[t_1, t_2]$	Average Velocity = $\frac{s(t_2)-s(t_1)}{t_2-t_1}$
$[0, 1]$	$\frac{s(1) - s(0)}{1 - 0} = \frac{-1 - 3}{1} = -4$
$[0, 2]$	$\frac{s(2) - s(0)}{2 - 0} = \frac{-1 - 3}{2} = -2$
$[0, 3]$	$\frac{s(3) - s(0)}{3 - 0} = \frac{3 - 3}{3} = 0$
$[0, 4]$	$\frac{s(4) - s(0)}{4 - 0} = \frac{11 - 3}{4} = 2$
$[3, 4]$	$\frac{s(4) - s(3)}{4 - 3} = \frac{11 - 3}{1} = 8$

On the general interval $[a, b]$ the Average Velocity is

$$\frac{s(b) - s(a)}{b - a} = \frac{2b^2 - 6b + 3 - (2a^2 - 6a + 3)}{b - a} = \frac{2(b^2 - a^2) - 6(b - a)}{b - a} =$$

$$\frac{2(b - a)(b + a) - 6(b - a)}{b - a} = 2(b + a) - 6$$

Let us look at the Average Velocity near $t_1 = 3$:

Time Interval = $[3, t_2]$	Average Velocity = $2(3 + t_2) - 6$
$[3, 4]$	$2(3 + 4) - 6 = 8$
$[3, 3.1]$	$2(3 + 3.1) - 6 = 6.2$
$[3, 3.01]$	$2(3 + 3.01) - 6 = 6.02$
$[3, 3.001]$	$2(3 + 3.001) - 6 = 6.002$
$[3, 3.0001]$	$2(3 + 3.0001) - 6 = 6.0002$
$[3, 3.00001]$	$2(3 + 3.00001) - 6 = 6.00002$

and

Time Interval = $[t_1, 3]$	Average Velocity = $2(t_1 + 3) - 6$
$[2, 3]$	$2(2 + 3) - 6 = 4$
$[2.9, 3]$	$2(2.9 + 3) - 6 = 5.8$
$[2.99, 3]$	$2(2.99 + 3) - 6 = 5.98$
$[2.999, 3]$	$2(2.999 + 3) - 6 = 5.998$
$[2.9999, 3]$	$2(2.9999 + 3) - 6 = 5.9998$
$[2.99999, 3]$	$2(2.99999 + 3) - 6 = 5.99998$

If we take the limit as t approaches 3, we get:

$$\lim_{t \to 3} \frac{s(t) - s(3)}{t - 3} = \lim_{t \to 3} 2(t + 3) - 6 = 2(3 + 3) - 6 = 12 - 6 = 6.$$

This limit is called the Instantaneous Velocity at time $t = 3$.

In general, the Instantaneous Velocity at time $t = a$ is defined to be

$$\lim_{t \to a} \frac{s(t) - s(a)}{t - a}$$

With our example function $s(t) = 2t^2 - 6t + 3$ we then have

$$\lim_{t \to a} \frac{s(t) - s(a)}{t - a} = \lim_{t \to a} 2(a + t) - 6 = 2(a + a) - 6 = 4a - 6$$

Another equivalent and very useful formula for the instantaneous velocity at time t is:

$$\lim_{h \to 0} \frac{s(t + h) - s(t)}{h}$$

With our example function $s(t) = 2t^2 - 6t + 3$ we then have

$$\lim_{h \to 0} \frac{s(t + h) - s(t)}{h} =$$
$$\lim_{h \to 0} \frac{2(t + h)^2 - 6(t + h) + 3 - (2t^2 - 6t + 3)}{h} =$$
$$\lim_{h \to 0} \frac{2(t^2 + 2th + h^2) - 6t - 6h + 3 - 2t^2 + 6t - 3}{h} =$$
$$\lim_{h \to 0} \frac{2t^2 + 4th + 2h^2 - 6t - 6h + 3 - 2t^2 + 6t - 3}{h} =$$
$$\lim_{h \to 0} \frac{4th + 2h^2 - 6h}{h} = \lim_{h \to 0} 4t + 2h - 6 = 4t + 2(0) - 6 = 4t - 6$$

Other Rates of Change

If a quantity $Q(t)$ varies with time, its average rate of change over the time interval $[t_0, t_1]$ is defined to be

$$\frac{Q(t_1) - Q(t_0)}{t_1 - t_0}$$

The instantaneous rate of change is defined to be

$$\lim_{t_1 \to t_0} \frac{Q(t_1) - Q(t_0)}{t_1 - t_0}$$

if this limit exists.

This limit is also equal to

$$\lim_{h \to 0} \frac{Q(t_0 + h) - Q(t_0)}{h}$$