

Techniques for Finding Derivatives

Derivative Rules:

(1) If c is a constant, then $\frac{d}{dx} [c] = 0$.

(2) If a is any real number, $\frac{d}{dx} [x^a] = ax^{a-1}$.

(3) If f is differentiable at x , $\frac{d}{dx} [cf(x)] = c \frac{d}{dx} [f(x)]$.

If f and g are differentiable at x , then

(4) $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$.

(5) $\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} [f(x)] - \frac{d}{dx} [g(x)]$.

These rules are sufficient for the differentiation of all polynomials.

Example 1: $\frac{d}{dx} [8x^{10}] = 8 \frac{d}{dx} [x^{10}] = 8 (10x^{10-1}) = 80x^9$

Example 2: $\frac{d}{dx} [x^5 + x^{-1}] = \frac{d}{dx} [x^5] + \frac{d}{dx} [x^{-1}] =$

$$5x^{5-1} + (-1)x^{-1-1} = 5x^4 - x^{-2}$$

Example 3: $\frac{d}{dx} [5x^4 - x^{-2}] = \frac{d}{dx} [5x^4] - \frac{d}{dx} [x^{-2}] =$

$$5 \frac{d}{dx} [x^4] - (-2)x^{-2-1} = 5(4)x^{4-1} + 2x^{-3} = 20x^3 + 2x^{-3}$$

The Product & Quotient Rules

The Product Rule

The derivative of the product of two non-constant functions is **never** equal to the product of their derivatives.

$$\text{We have } \frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx} [g(x)] + g(x) \frac{d}{dx} [f(x)]$$

or, in function notation

$$(fg)' = fg' + f'g$$

or

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$$

Example 4: Let $f(x) = x$, and $g(x) = x$, so that $f(x)g(x) = x \cdot x = x^2$. Then the Product Rule gives us

$$(x \cdot x)' = x(x)' + (x)'x = x(1) + (1)x = 2x, \quad \text{which agrees with } (x^2)' = 2x.$$

Notice that $f'(x)g'(x) = 1(1) = 1$, which is quite different, and incorrect.

Example 5: Find the derivative of $(2x^2 + 3)(4x + 5)$.

Let $f(x) = 2x^2 + 3$, and $g(x) = 4x + 5$.

Then $f'(x) = 4x$ and $g'(x) = 4$, so $f(x)g(x) = (2x^2 + 3)(4x + 5)$, and

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x) = (2x^2 + 3)(4) + (4x)(4x + 5) = (8x^2 + 12) + (16x^2 + 20x) = 24x^2 + 20x + 12$$

We often write this calculation in a shorter form:

$$(2x^2 + 3)(4x + 5)' + (2x^2 + 3)'(4x + 5) = (2x^2 + 3)(4) + (4x)(4x + 5) = 24x^2 + 20x + 12$$

Example 6: Find the derivative of $(x^2 + 3x)(x^2 + 2)$.

Let $f(x) = x^2 + 3x$, and $g(x) = x^2 + 2$.

Then $f'(x) = 2x + 3$ and $g'(x) = 2x$, so

$$f(x)g(x) = (x^2 + 3x)(x^2 + 2), \text{ and}$$

$$\begin{aligned}(f(x)g(x))' &= f(x)g'(x) + f'(x)g(x) = \\ &(x^2 + 3x)(2x) + (2x + 3)(x^2 + 2) = \\ &2x^3 + 6x^2 + 2x^3 + 4x + 3x^2 + 6 = \\ &4x^3 + 9x^2 + 4x + 6\end{aligned}$$

We write this in a shorter form:

$$(x^2 + 3x)(x^2 + 2)' + (x^2 + 3x)'(x^2 + 2) =$$

$$(x^2 + 3x)(2x) + (2x + 3)(x^2 + 2) =$$

$$\frac{2x^3 + 6x^2 + 2x^3 + 4x + 3x^2 + 6}{4x^3 + 9x^2 + 4x + 6} =$$

The Quotient Rule

The derivative of the quotient of two non-constant functions requires an even more complicated formula:

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{[g(x)]^2}$$

or, in function notation,

$$\left(\frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}$$

or

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example 7: Let $f(x) = x + 1$, and $g(x) = x - 1$, so that $\frac{f(x)}{g(x)} = \frac{x + 1}{x - 1}$. Then the Quotient Rule gives us:

$$\left(\frac{x + 1}{x - 1} \right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} =$$

$$\frac{(x - 1)(x + 1)' - (x + 1)(x - 1)'}{(x - 1)^2} =$$

$$\frac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2} =$$

$$\frac{(x - 1) - (x + 1)}{(x - 1)^2} =$$

$$\frac{x - 1 - x - 1}{(x - 1)^2} = \frac{-2}{(x - 1)^2}.$$

Notice that $\frac{f'(x)}{g'(x)} = \frac{1}{1} = 1$, which is quite different, and incorrect.

Example 8: Differentiate $\frac{x^2 - 4x + 2}{x + 3}$

Solution: $\left(\frac{x^2 - 4x + 2}{x + 3}\right)' =$

$$\frac{(x + 3)(x^2 - 4x + 2)' - (x^2 - 4x + 2)(x + 3)'}{(x + 3)^2} =$$

$$\frac{(x + 3)(2x - 4) - (x^2 - 4x + 2)(1)}{(x + 3)^2} =$$

$$\frac{2x^2 + 2x - 12 - x^2 + 4x - 2}{(x + 3)^2} = \frac{x^2 + 6x - 14}{(x + 3)^2}$$

Example 9: Differentiate $\frac{\sqrt{t}}{2t + 3}$

Solution: $\left(\frac{\sqrt{t}}{2t + 3}\right)' = \left(\frac{t^{\frac{1}{2}}}{2t + 3}\right)'$

$$\frac{(2t + 3)(t^{\frac{1}{2}})' - (t^{\frac{1}{2}})(2t + 3)'}{(2t + 3)^2} =$$

$$\frac{(2t + 3)\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}(2)}{(2t + 3)^2} = \frac{(2t + 3)\frac{1}{2\sqrt{t}} - 2\sqrt{t}}{(2t + 3)^2} =$$

$$\frac{\frac{t}{\sqrt{t}} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\sqrt{t} + \frac{3}{2\sqrt{t}} - 2\sqrt{t}}{(2t+3)^2} = \frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2} =$$

$$\frac{\frac{3}{2\sqrt{t}} - \sqrt{t}}{(2t+3)^2} \left(\frac{2\sqrt{t}}{2\sqrt{t}} \right) = \frac{3 - 2t}{2\sqrt{t}(2t+3)^2}$$
