

Derivatives

Definition: If the limit

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists, it is called the **derivative** of f at a ,

and is denoted by $f'(a)$,

pronounced “ f prime of a .”

Example 1: $f(x) = x^2$.

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{(a+h)^2 - a^2}{h} =$$

$$\lim_{h \rightarrow 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h \rightarrow 0} \frac{2ah + h^2}{h} = \lim_{h \rightarrow 0} \frac{(2a+h)h}{h} =$$

$$\lim_{h \rightarrow 0} 2a + h = 2a + 0 = 2a$$

Note: This tells us that the slope of the tangent line to the parabola $y = x^2$ at any point (x, x^2) is always twice the x -coordinate.

Example 2: $f(x) = x^3$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} =$$

$$\lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} =$$

$$\lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{(3x^2 + 3xh + h^2)h}{h} =$$

$$\lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2 + 3x(0) + 0^2 = 3x^2$$

Example 3: Find the equation of the tangent line to $y = f(x) = 5x^2 - 3x$, at $(2, 14)$,

$$f(a) = f(2) = 5(2)^2 - 3(2) = 5(4) - 6 = 20 - 6 = 14$$

$$m_{PQ} = \frac{f(a+h) - f(a)}{h} = \frac{f(2+h) - f(2)}{h} = \frac{5(2+h)^2 - 3(2+h) - 14}{h} =$$

$$\frac{5(4 + 4h + h^2) - 6 - 3h - 14}{h} = \frac{20 + 20h + 5h^2 - 6 - 3h - 14}{h} =$$

$$\frac{20h + 5h^2 - 3h}{h} = \frac{17h + 5h^2}{h} = 17 + 5h$$

Taking limits,

$$m = \lim_{h \rightarrow 0} m_{PQ} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} 17 + 5h = 17 + 5(0) = 17$$

Therefore the equation of the tangent line is

$$y - 14 = 17(x - 2)$$

Example 4: $f(x) = \frac{1}{x}$.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} =$$

$$\lim_{h \rightarrow 0} \frac{\frac{1}{x+h} \left(\frac{x}{x}\right) - \frac{1}{x} \left(\frac{x+h}{x+h}\right)}{h} =$$

$$\lim_{h \rightarrow 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} = \lim_{h \rightarrow 0} \frac{x - (x+h)}{x(x+h)h} =$$

$$\lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{\frac{1}{h}} = \lim_{h \rightarrow 0} \left(\frac{-h}{x(x+h)} \right) \left(\frac{1}{h} \right) = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} =$$

$$\frac{-1}{x(x+0)} = \frac{-1}{x^2} \text{ if } x \neq 0.$$

Remember: $\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b} \right) \left(\frac{d}{c} \right)$

The Derivative as a Function

The slope defines a new function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ called the derivative of } f.$$

Since the limit doesn't necessarily exist, the derivative may have a smaller domain than f .

Definition: We define the derivative of f to be the function f' defined by the rule

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists.

Example 5: If $f(x) = x^4 - x^2$, find a formula for $f'(x)$.

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^4 - (x+h)^2] - [x^4 - x^2]}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{[x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^2 - 2xh - h^2] - x^4 + x^2}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2xh - h^2}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{(4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h)h}{h} = \\
 &= \lim_{h \rightarrow 0} 4x^3 + 6x^2h + 4xh^2 + h^3 - 2x - h = \mathbf{4x^3 - 2x}
 \end{aligned}$$

Other Notations

The derivative of $y = f(x)$ is written in many different ways in the many branches of Science that use derivatives:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} (f(x)) = D_x y = D_x (f(x))$$

Some Useful Theory

Definition: If the derivative of f exists at x , we say that f is **differentiable** at x . f is **differentiable on an open interval** (a, b) if it is differentiable at every $x \in (a, b)$.

We have already observed that a limit of a quotient where the denominator approaches zero cannot exist unless the numerator also approaches zero.

In the case of the limit of a quotient that we use to define the derivative function, the denominator h always approaches zero, so the numerator must also approach zero if the derivative is to exist.

In other words, if the derivative exists, we have

$$\lim_{h \rightarrow 0} f(x + h) - f(x) = 0, \text{ or}$$

$$\lim_{h \rightarrow 0} f(x + h) = \lim_{h \rightarrow 0} f(x) = f(x), \text{ so we have the}$$

Theorem: If f is differentiable at x , it is also continuous at x .

Note that the converse is not always true: try $f(x) = |x|$ at $x = 0$.

What Goes On When Differentiability Fails?

First, there might be a **discontinuity** of the function. If the function is continuous, then the left- or right-hand limits

$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h}$ and $\lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$ may be

unequal, be infinite, or just plain fail to exist in any sense at all.