The Chain Rule

Recall that the composite function or composition of two functions is the function obtained by applying them one after the other.

For example, if $f(x) = \frac{1}{x}$ and $g(x) = x^3 + 2$, then

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{x^3 + 2}$$

and $g(f(x)) = (f(x))^3 + 2 = \left(\frac{1}{x}\right)^3 + 2 = \frac{1}{x^3} + 2$

It is important for the student to be able to recognize that a function is the composition of two or more functions, so that the upcoming "Chain Rule" for differentiating composites may be used.

For example, if $h(x) = \dfrac{1}{x^3 + 2}$, we see that we can let $u = g(x) = x^3 + 2$ to get $h(x) = \dfrac{1}{u} = \dfrac{1}{g(x)}$.

Then we let $f(x) = \dfrac{1}{x}$, so that $h(x) = f(u) = f(g(x))$

The derivative of the composition of two non-constant functions is equal to the product of their derivatives, evaluated appropriately.

We have, in function notation,

$$\left(g(h(x))\right)' = g'(h(x))h'(x)$$

or, in Leibnitz notation, if $y = f(g(x))$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We may also write

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Example 1: Using $g(x) = \frac{1}{x} = x^{-1}$ and $h(x) = x^3 + 2$,

we have $g'(x) = (-1)x^{-2}$ and $h'(x) = 3x^2$,

$g'(h(x)) = (-1)(h(x))^{-2}$, so we get

$$\left(\frac{1}{x^3+2}\right)' = g'(h(x))h'(x) = (-1)(h(x))^{-2}(3x^2) =$$

$$(-1)(x^3+2)^{-2}(3x^2) = \frac{-3x^2}{(x^3+2)^2}$$

Example 2: Let $g(x) = x^3$, and $h(x) = x^2$, so that

$g(h(x)) = (h(x))^3 = (x^2)^3 = x^6$.

Then $g'(x) = 3x^2$, so $g'(h(x)) = 3(h(x))^2$, and $h'(x) = 2x$,

so the Chain Rule gives us

$(g(h(x)))' = g'(h(x))h'(x) = \left(3(h(x))^2\right)(2x) =$

$\left(3(x^2)^2\right)(2x) = \left(3x^4\right)(2x) =$ $6x^5$, as expected.

Example 3: Let $g(x) = x^3 + 3$, and $h(x) = x^2 + 2$, so that

$g(h(x)) = (h(x))^3 + 3 = (x^2 + 2)^3 + 3.$

Then $g'(x) = 3x^2$, so $g'(h(x)) = 3(h(x))^2$, and $h'(x) = 2x$,

so the Chain Rule gives us

$$(g(h(x)))' = g'(h(x))h'(x) = \left(3(h(x))^2\right)(2x) =$$
$$\left(3(x^2 + 2)^2\right)(2x) =$$

$6x(x^2 + 2)^2$

Example 4: Find $f'(x)$ if $f(x) = \sqrt[3]{x^4 + x^2 + 1}$.

We let $g(x) = x^{\frac{1}{3}}$ and $u = x^4 + x^2 + 1$ so that $f(x) = g(u)$.

Then $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, $g'(u) = \frac{1}{3}u^{-\frac{2}{3}}$, and $\dfrac{du}{dx} = 4x^3 + 2x$,

so we have $f'(x) = g'(u)\dfrac{du}{dx} = \dfrac{1}{3}(u)^{-\frac{2}{3}}(4x^3 + 2x) =$

$$\frac{2x(2x^2 + 1)}{3\left(x^4 + x^2 + 1\right)^{\frac{2}{3}}}$$

The Power Rule

$$\left((g(x))^n\right)' = n\,(g(x))^{n-1}\,g'(x)$$

Example 4a: Find $f'(x)$ if $f(x) = \sqrt[3]{x^4 + x^2 + 1}$.

We write $f(x) = (g(x))^{\frac{1}{3}}$ where $g(x) = x^4 + x^2 + 1$.

Then

$$f'(x) = \frac{1}{3}(g(x))^{-\frac{2}{3}}g'(x) = \frac{1}{3}\left(x^4 + x^2 + 1\right)^{-\frac{2}{3}}(4x^3 + 2x) =$$

$$\frac{2x(2x^2 + 1)}{3\left(x^4 + x^2 + 1\right)^{\frac{2}{3}}}$$

Example 5: Find $f'(x)$ if $f(x) = \left(\frac{4x-3}{2x+1}\right)^8$.

We have $f'(x) = 8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{4x-3}{2x+1}\right)' =$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{(2x+1)(4x-3)' - (4x-3)(2x+1)'}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{(2x+1)4 - (4x-3)2}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{8x+4-8x+6}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7 \left(\frac{10}{(2x+1)^2}\right) = \boxed{80\frac{(4x-3)^7}{(2x+1)^9}}$$

Example 6: Find $f'(x)$ if $f(x) = \dfrac{(4x-3)^8}{(2x+1)^5}$.

First we use the Quotient Rule:

$$f'(x) = \frac{[(2x+1)^5]\,[(4x-3)^8]' - (4x-3)^8\,[(2x+1)^5]'}{[(2x+1)^5]^2}$$

and then the Power Rule:

$$f'(x) =$$

$$\frac{[(2x+1)^5]\,8(4x-3)^{8-1}(4x-3)' - (4x-3)^8 5(2x+1)^{5-1}(2x+1)'}{(2x+1)^{5\times 2}} =$$

$$\frac{(2x+1)^5 8(4x-3)^7(4) - (4x-3)^8 5(2x+1)^4(2)}{(2x+1)^{10}} =$$

$$\frac{32(2x+1)^5(4x-3)^7 - 10(4x-3)^8(2x+1)^4}{(2x+1)^{10}} =$$

$$\frac{32(2x+1)(4x-3)^7 - 10(4x-3)^8}{(2x+1)^6} =$$

$$(4x-3)^7\,\frac{32(2x+1) - 10(4x-3)}{(2x+1)^6} =$$

$$\frac{(4x-3)^7}{(2x+1)^6}\left[64x + 32 - 40x + 30\right] = \frac{(4x-3)^7}{(2x+1)^6}\left[24x + 62\right] =$$

$$2\,\frac{(4x-3)^7}{(2x+1)^6}\left[12x + 31\right]$$