Factoring

As we have already seen, the skill of factoring integers is crucial to our mastery of the arithmetic of fractions. The skill of factoring polynomials will be crucial to mastering the Calculus of polynomials, and quotients of polynomials — which are usually called rational functions.

Examples of factorizations of polynomials:

$$2x^2 + x - 1 = (2x - 1)(x + 1)$$

$$8x^2 - 2x - 15 = (2x - 3)(4x + 5)$$

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

$$2x^3 - \frac{17}{3}x^2 - \frac{5}{3}x + 2 = (x - 3)(2x - 1)\left(x + \frac{2}{3}\right)$$

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

$$x^4 - 16 = (x - 2)(x + 2)(x^2 + 4)$$

Factoring a polynomial is not always a pleasant or easy operation, but it is important: because in addition to telling us where a polynomial is zero, it is the tool we will need to determine exactly where a polynomial is positive or negative. This in turn will vital to the sketching of the graphs of polynomials, one of the highlights of this course.

We will study the procedure in some depth, first for quadratic (or degree 2) polynomials, and then for general polynomials.

Factoring Quadratic Terms

Sometimes the factorization of a quadratic can be arrived at by "inspection":

Example 1: Factor the quadratic $x^2 + x - 6$.

Solution: If we assume that $x^2 + x - 6$ factors into linear terms with *integer* coefficients, then we would have:
$x^2 + x - 6 = (x + a)(x + b)$
where a and b are two integers which need to be determined. Multiplying out the right-hand side of this equation gives:

$x^2 + x - 6 = x^2 + (a + b)x + ab$

and so we must choose a and b so that $a + b = 1$ and $ab = -6$. The second condition gives you a small number of *integer* possibilities to test, i.e., those pairs of integers whose product equals -6

a	-6	-3	-2	-1	1	2	3	6
$b = \frac{-6}{a}$	1	2	3	6	-6	-3	-2	-1
$a + b$	-5	-1	1	5	-5	-1	1	5

so we can either take $a = -2$ and $b = 3$ or $a = 3$ and $b = -2$. Either way, we get the factorizations $x^2 + x - 6 = (x + 2)(x - 3) = (x - 3)(x + 2)$.

Example 2: Factor the quadratic $2x^2 + x - 6$.

Solution: If we assume that $2x^2 + x - 6$ factors into linear terms with *integer* coefficients, then we would have:
$2x^2 + x - 6 = (2x + a)(x + b)$
where $2x$ and x are necessary to obtain the $2x^2$ term, and a and b are two integers which need to be determined. Multiplying out the right-hand side of this equation gives:

$2x^2 + x - 6 = 2x^2 + (a + 2b)x + ab$

and so we must choose a and b so that $a + 2b = 1$ and $ab = -6$. The second condition gives you a small number of *integer* possibilities to test, i.e., those pairs of integers whose product equals -6

a	-6	-3	-2	-1	1	2	3	6
$b = \frac{-6}{a}$	1	2	3	6	-6	-3	-2	-1
$a + 2b$	-4	1	4	11	-11	-4	-1	4

We see that only $a = -3$ and $b = 2$ satisfy the condition $a + 2b = 1$, so we have the factorization:
$2x^2 + x - 6 = (2x + (-3))(x + 2) = (2x - 3)(x + 2)$

Factorization is related to the *roots* of polynomials: a number r is a root of $p(x)$ if $p(r) = 0$. If a quadratic polynomial can be factored as

$$ax^2 + bx + c = a(x - r_1)(x - r_2)$$

then $p(r_1) = p(r_2) = 0$, so that r_1 and r_2 are roots of $p(x)$. The reverse is true, so that factoring a polynomial is equivalent to finding its roots. This is fortunately made easy by the:

Quadratic Formula

The roots r_1 and r_2 of the quadratic term $ax^2 + bx + c$ are equal to $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Thus our two roots r_1 and r_2 are given by

$r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

and thus we have the factorization
$ax^2 + bx + c = a\left(x - \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

Example 2a: Factor the quadratic $2x^2 + x - 6$ using the Quadratic Formula.

Solution: Using the quadratic formula with $a = 2$, $b = 1$, and $c = -6$, the roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-1 + \sqrt{1^2 - 4(2)(-6)}}{2(2)} =$$

$$\frac{-1 + \sqrt{1 + 48}}{4} = \frac{-1 + \sqrt{49}}{4} = \frac{-1 + 7}{4} = \frac{3}{2}$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-1 - 7}{4} = -\frac{8}{4} = -2$$

Thus $2x^2 + x - 6 = 2(x - (-2))\left(x - \frac{3}{2}\right)$

which appears to be different from the factorization $2x^2 + x - 6 = (2x + (-3))(x + 2) = (2x - 3)(x + 2)$ that we found by "inspection"! However, we note that

$$2(x - (-2))\left(x - \frac{3}{2}\right) = (x + 2)2\left(x - \frac{3}{2}\right) =$$

$$(x + 2)\left(2x - 2\frac{3}{2}\right) = (x + 2)(2x - 3) = (2x - 3)(x + 2),$$

so we have two different but equivalent factorizations of the same polynomial.

Example 3 Factor the quadratic $3x^2 - 5x + 1$

Solution: Using the quadratic formula with $a = 3$, $b = -5$, and $c = 1$, the roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} =$$

$$\frac{-(-5) + \sqrt{(-5)^2 - 4(3)(1)}}{2(3)} = \frac{5 + \sqrt{25 - 12}}{6} = \frac{5 + \sqrt{13}}{6}$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{5 - \sqrt{13}}{6}$$

Thus $3x^2 - 5x + 1 = 3\left(x - \frac{5 + \sqrt{13}}{6}\right)\left(x - \frac{5 - \sqrt{13}}{6}\right)$

Check:

You can (and perhaps *should*) always check a factorization by multiplying the terms to see if you get what you started with.

$$3x^2 - 5x + 1 = 3\left(x - \frac{5 + \sqrt{13}}{6}\right)\left(x - \frac{5 - \sqrt{13}}{6}\right) =$$

$$3\left[x^2 - \left(\frac{5 + \sqrt{13}}{6} + \frac{5 - \sqrt{13}}{6}\right)x + \left(\frac{5 + \sqrt{13}}{6}\right)\left(\frac{5 - \sqrt{13}}{6}\right)\right] =$$

$$3\left[x^2 - \left(\frac{5}{6} + \frac{5}{6}\right)x + \left(\frac{5^2 - \left(\sqrt{13}\right)^2}{6^2}\right)\right] =$$

$$3\left[x^2 - \frac{5}{3}x + \left(\frac{25 - 13}{36}\right)\right] = 3\left[x^2 - \frac{5}{3}x + \frac{12}{36}\right] = 3\left[x^2 - \frac{5}{3}x + \frac{1}{3}\right] = 3x^2 - 5x + 1$$

Example 4 Factor the quadratic $x^2 - 2x + 2$

Solution:

Using the quadratic formula with $a = 1$, $b = -2$, and $c = 2$, the roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} =$$

$$\frac{-(-2) + \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 + \sqrt{4 - 8}}{2} = \frac{2 + \sqrt{-4}}{2} = 1 + i$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) - \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 - \sqrt{4-8}}{2} = \frac{2 - \sqrt{-4}}{2} = 1 - i$$

where i is the "imaginary" number $\sqrt{-1}$.

Thus $x^2 - 2x + 2 = (x - (1 + i))\,(x - (1 - i))$

This is perfectly correct, but it is *not* a factorization into linear factors with *real terms*. Since in elementary Calculus we don't deal with imaginary factors, we will say that $x^2 - 2x + 2$ and any other polynomial with imaginary roots cannot be factored. We sometimes say that they are irreducible.

Factoring General Polynomials

It is a theorem of algebra that any polynomial can be factored into real and irreducible quadratic factors. *It is, however, entirely another matter to actually determine what these factors are!* Generally the factorization is done in stages: factor it into two polynomials of lower degree, and then try to factor each of them.

Three very important factorizations to remember are:

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Example 5 Factor the cubic $8x^3 - 27$

Solution: You should recognize this as a difference of cubes, i.e.,

$8x^3 - 27 = (2x)^3 - 3^3 = (2x - 3)\left[(2x)^2 + (2x)(3) + 3^2\right] = (2x-3)(4x^2 + 6x + 9)$

Can we factor the quadratic factor $4x^2 + 6x + 9$? We apply the quadratic formula to it and find the roots are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(4)(9}}{2(4)} = \frac{-6 \pm \sqrt{36 - 144}}{8} = \frac{-6 \pm \sqrt{-108}}{8}$$

which are imaginary, so $4x^2 + 6x + 9$ is irreducible. Thus our factorization is as complete as possible

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Example 6: Factor the 5^{th} degree polynomial $x^5 - 16x$

Solution: If a polynomial has no constant term, then you have one factor for free(!), namely x. Thus

$$x^5 - 16x = x(x^4 - 16) = x\left((x^2)^2 - 4^2\right) =$$

$$x\left(x^2 - 4\right)\left(x^2 + 4\right) = x(x-2)(x+2)(x^2+4)$$

Note that $x^2 + 4$ is irreducible because it has no real roots.

The Factor Theorem

The linear polynomial $x - r$ is a factor of a polynomial $p(x)$ if and only if r is a root of $p(x)$, i.e., $p(r) = 0$

This is very much like the quadratic case, the big difference is that there is no formula for the roots when the degree is greater than 4!

Example 7: Factor the third degree polynomial
$p(x) = x^3 - 5x^2 + 6x - 2$.

Solution: If this polynomial has any integer roots they must divide 2. Since the only integers which divide 2 are -2, -1, 1, and 2, it is useful to construct a table of values:

x	-2	-1	1	2
$p(x)$	-42	-14	0	-2

so the only integer root is 1. We divide $x - 1$ into $x^3 - 5x^2 + 6x - 2$:

$$
\begin{array}{r|rrrr}
 & & x^2 & -4x & +2 \\ \hline
x-1 & x^3 & -5x^2 & +6x & -2 \\
 & x^3 & -x^2 & & \\ \hline
 & & -4x^2 & +6x & \\
 & & -4x^2 & +4x & \\ \hline
 & & & 2x & -2 \\
 & & & 2x & -2 \\ \hline
 & & & & 0
\end{array}
$$

so we have $x^3 - 5x^2 + 6x - 2 = (x-1)(x^2 - 4x + 2)$ and now we find the roots of the quadratic $x^2 - 4x + 2$ using the quadratic formula with $a = 1$, $b = -4$, $c = 2$:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) + \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 + \sqrt{16-8}}{2} = \frac{4 + \sqrt{8}}{2} = \frac{4 + 2\sqrt{2}}{2} = 2 + \sqrt{2},$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) - \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 - \sqrt{16-8}}{2} = \frac{4 - \sqrt{8}}{2} = \frac{4 - 2\sqrt{2}}{2} = 2 - \sqrt{2},$$

so the complete factorization of $p(x)$ is

$$p(x) = (x-1)\left(x - (2 + \sqrt{2})\right)\left(x - (2 - \sqrt{2})\right)$$

The Rational Root Test

Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with *integer* coefficients, and $r = \dfrac{m}{q}$ is a rational number expressed in lowest terms.
Then $r = \dfrac{m}{q}$ can be a root of $p(x)$ only if m divides the constant term a_0 and q divides the leading coefficient a_n.

Notice that this only says that $\frac{m}{q}$ *can* be a root, it does not say that it *is*!

Example 8: Factor the third degree polynomial
$p(x) = 3x^3 - 8x^2 + x + 2$.

Solution: Suppose $\frac{m}{q}$ is a root. Then m divides 2 and q divides 3, so the possibilities are $m = -2, -1, 1, 2$, and $q = -3, -1, 1$, and 3, leaving us with 16 possible values for $\frac{m}{q}$, as shown in the following table.

	$m = -2$	$m = -1$	$m = 1$	$m = 2$
$q = -3$	$\frac{m}{q} = \frac{-2}{-3} = \frac{2}{3}$	$\frac{m}{q} = \frac{-1}{-3} = \frac{1}{3}$	$\frac{m}{q} = \frac{1}{-3} = -\frac{1}{3}$	$\frac{m}{q} = \frac{2}{-3} = -\frac{2}{3}$
$q = -1$	$\frac{m}{q} = \frac{-2}{-1} = 2$	$\frac{m}{q} = \frac{-1}{-1} = 1$	$\frac{m}{q} = \frac{1}{-1} = -1$	$\frac{m}{q} = \frac{2}{-1} = -2$
$q = 1$	$\frac{m}{q} = \frac{-2}{1} = -2$	$\frac{m}{q} = \frac{-1}{1} = -1$	$\frac{m}{q} = \frac{1}{1} = 1$	$\frac{m}{q} = \frac{2}{1} = 2$
$q = 3$	$\frac{m}{q} = \frac{-2}{3} = -\frac{2}{3}$	$\frac{m}{q} = \frac{-1}{3} = -\frac{1}{3}$	$\frac{m}{q} = \frac{1}{3}$	$\frac{m}{q} = \frac{2}{3}$

But there is duplication, so the actual possible values are: $\frac{m}{q} = -2, -1, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 1, 2$. We plug them into the polynomial:

x	-2	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	2
$p(x)$	-56	-4	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{9}$	0	-2	-4

We see that $\frac{2}{3}$ is the only rational root. In particular, $x - \frac{2}{3}$ is a factor of $p(x)$. We could perform the long division of $x - \frac{2}{3}$ into $3x^3 - 8x^2 + x + 2$, but this would lead to some messy calculations with fractions that can easily be avoided by noticing that $x - \frac{2}{3} = \frac{1}{3}(3x - 2)$. We divide $3x - 2$ into $p(x)$:

$$
\begin{array}{r|rrrr}
 & & x^2 & -2x & -1 \\ \hline
3x-2 & 3x^3 & -8x^2 & +x & +2 \\
 & 3x^3 & -2x^2 & & \\ \hline
 & & -6x^2 & +x & \\
 & & -6x^2 & +4x & \\ \hline
 & & & -3x & +2 \\
 & & & -3x & +2 \\ \hline
 & & & & 0
\end{array}
$$

So now we have $p(x) = 3x^3 - 8x^2 + x + 2 = (3x-2)(x^2 - 2x - 1) = 3\left(x - \frac{2}{3}\right)(x^2 - 2x - 1)$.

We now factor the quadratic factor $x^2 - 2x - 1$ by using the quadratic formula with $a = 1$, $b = -2$, and $c = -1$ to find its roots:

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \dfrac{2 \pm \sqrt{4+4}}{2} = \dfrac{2 \pm \sqrt{8}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$, so we have $x^2 - 2x - 1 = \left((x - (1 + \sqrt{2})\right)\left((x - (1\right.$

and therefore

$p(x) = (3x-2)\left(x - (1+\sqrt{2})\right)\left(x - (1-\sqrt{2})\right) = 3\left(x - \frac{2}{3}\right)\left(x - (1+\sqrt{2})\right)\left(x - (1-\sqrt{2})\right)$