Arithmetic of Fractions
Multiplication

It is very easy to multiply two fractions together: simply multiply together the numerators and denominators:

$$\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

or, using the dot notation for multiplication:

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Example 1: $\left(\frac{5}{8}\right) \times \left(\frac{7}{3}\right) = \frac{5 \times 7}{8 \times 3} = \frac{35}{24}$

Suppose $c = x$ and $d = 1$, so that $\frac{c}{d} = \frac{x}{1} = x$. Then $\left(\frac{a}{b}\right) \cdot x = \left(\frac{a}{b}\right) \cdot \left(\frac{x}{1}\right) = \frac{ax}{b(1)} = \frac{ax}{b}$,

so multiplying a fraction by x has the same effect as multiplying the numerator of the fraction by x.

Example 2: $\frac{2}{3}\pi = \frac{2\pi}{3}$

Example 3: Combine $\left(\frac{7}{5}\right) \cdot \left(\frac{-3}{8}\right)$ into one fraction.

Solution: $\left(\frac{7}{5}\right)\cdot\left(\frac{-3}{8}\right) = \frac{(7)(-3)}{5(8)} = \quad -\frac{21}{40}$

Example 4: Combine $\left(\frac{10}{3}\right)\cdot\left(\frac{9}{4}\right)$ into one fraction.

Solution: $\left(\frac{10}{3}\right)\cdot\left(\frac{9}{4}\right)$ =(factoring 10, 9, and 4, and cancelling)

$\frac{(2\times 5)(3\times 3)]}{3(2\times 2)} = \frac{(5)(3)}{2} = \quad \frac{15}{2}$

Addition

Case 1: Same Denominator:

To add two fractions with the same denominator, *add the numerators together*, i.e.,

$$\frac{a}{h}+\frac{b}{h}=\frac{a+b}{h}$$

(Consider: two-sixths of a pizza plus one-sixth of a pizza equals three-sixths, or one-half of a pizza.)

Case 2: Different Denominators:

For fractions with different denominators, say $\frac{a}{b}$ and $\frac{c}{d}$, we must first alter the fractions to have a **common denominator**, say $h = bd$. We do this by multiplications by 1, where we write 1 as an appropriate fraction:

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot 1+1\cdot\frac{c}{d}=\left(\frac{a}{b}\right)\cdot\left(\frac{d}{d}\right)+\left(\frac{b}{b}\right)\cdot\left(\frac{c}{d}\right)=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

We have thus derived the basic addition formula:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

The formula is easily remembered as "cross-multiplication":

$$\frac{a}{b} \bowtie \frac{c}{d} \to \frac{ad+bc}{bd}$$

Example 5:

$$\frac{3}{5}+\frac{7}{8}=\left(\frac{3}{5}\right)\left(\frac{8}{8}\right)+\left(\frac{7}{8}\right)\left(\frac{5}{5}\right)=\frac{24}{40}+\frac{35}{40}= \quad \frac{59}{40}$$

Example 6:

$$\frac{2}{3}+\frac{6}{5}=\left(\frac{2}{3}\right)\left(\frac{5}{5}\right)+\left(\frac{6}{5}\right)\left(\frac{3}{3}\right)=\frac{10}{15}+\frac{18}{15}= \quad \frac{28}{15}$$

Factor if You Can

Calculations with fractions can be simplified by factoring if there are common factors:

Example 7:

$$\frac{15}{28}+\frac{24}{35}=\frac{3\cdot 5}{4\cdot 7}+\frac{3\cdot 8}{5\cdot 7}=\left(\frac{3}{7}\right)\left(\frac{5}{4}\right)+\left(\frac{3}{7}\right)\left(\frac{8}{5}\right)=\left(\frac{3}{7}\right)\left(\frac{5}{4}+\frac{8}{5}\right)=\left(\frac{3}{7}\right)\left(\frac{5\cdot 5+8\cdot 4}{4\cdot 5}\right)=$$

$$\left(\frac{3}{7}\right)\left(\frac{25+32}{20}\right)=\left(\frac{3}{7}\right)\left(\frac{57}{20}\right)= \quad \frac{171}{140}$$

which would normally be done with fewer steps displayed:

$$\frac{15}{28}+\frac{24}{35}=\left(\frac{3}{7}\right)\left(\frac{5}{4}+\frac{8}{5}\right)=\left(\frac{3}{7}\right)\left(\frac{25+32}{20}\right)=\left(\frac{3}{7}\right)\left(\frac{57}{20}\right)= \quad \frac{171}{140}$$

Note that the straightforward approach is much more cumbersome:

$$\frac{15}{28}+\frac{24}{35}=\frac{(15)(35)+(28)(24)}{(28)(35)}=\frac{525+672}{980}=\frac{525+672}{980}=\frac{1197}{980}=\frac{7(171)}{7(140)}=\boxed{\frac{171}{140}}$$

and this only works if we know that 7 is a common factor of 1197 and 980. Not many of us have this information at our finger tips!

Danger: Notice that $\frac{a}{b}+\frac{c}{d}$ does **NOT** equal $\frac{a+c}{b+d}$!!

If in doubt, try $\frac{1}{2}+\frac{1}{2}?=?\frac{1+1}{2+2}=\frac{2}{4}=\frac{1}{2}$, and you know that two halves make a whole, not a half!

As a special case, consider a fraction $\frac{c}{d}$ added to an arbitrary number x, which we write as the fraction $\frac{x}{1}$. Then

$$x+\frac{c}{d}=\frac{x}{1}+\frac{c}{d}=\frac{xd+1c}{1d}=\frac{xd+c}{d}$$

Example 8: Combine $\frac{5}{4}+3$ into one fraction.

Solution

$$\frac{5}{4}+3=\frac{5}{4}+3\frac{4}{4}=\frac{5}{4}+\frac{12}{4}=$$

$$\boxed{\frac{17}{4}}$$

Subtraction

If you know how to add fractions then you know how to subtract them; the trick is to convert $-\left(\frac{c}{d}\right)$ into $\frac{-c}{d}$:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad + b(-c)}{bd} = \frac{ad - bc}{bd},$$

so we have the subtraction formula $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

Example 9:

$$\frac{5}{7} - \frac{3}{4} = \frac{5}{7} + \frac{-3}{4} = \frac{5(4) + 7(-3)}{7(4)} = \frac{20 - 21}{28} = -\frac{1}{28}$$

Factorization can again be used to simplify subtraction calculations:

Example 10:

$$\frac{5}{12} - \frac{3}{28} = \frac{5}{3(4)} - \frac{3}{4(7)} = \frac{1}{4}\left(\frac{5}{3} - \frac{3}{7}\right) = \frac{1}{4}\left(\frac{5(7) - 3(3)}{3(7)}\right) =$$

$$\frac{1}{4}\left(\frac{35 - 9}{21}\right) = \frac{1}{4}\left(\frac{26}{21}\right) = \frac{1}{2}\left(\frac{13}{21}\right) = \frac{13}{42}$$

Division

To divide $\frac{a}{b}$ by $\frac{c}{d}$, we remember that $\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{cd} = 1$,

and multiply by 1 in a special form:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \cdot \frac{\left(\frac{d}{c}\right)}{\left(\frac{d}{c}\right)} = \frac{\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)}{\left(\frac{c}{d}\right)\left(\frac{d}{c}\right)} = \frac{\left(\frac{ad}{bc}\right)}{1} = \frac{ad}{bc}$$

Our basic division formula is thus

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$$

Example 11: $\frac{\frac{2}{3}}{\frac{5}{4}} = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \quad \frac{8}{15}$

Example 12: $\frac{\frac{2}{3}+\frac{4}{7}}{\frac{5}{7}+\frac{3}{4}} = \frac{\frac{14+12}{21}}{\frac{20+21}{28}} = \frac{\frac{26}{21}}{\frac{41}{28}} = \left(\frac{26}{21}\right)\left(\frac{28}{41}\right) = \left(\frac{26}{3}\right)\left(\frac{4}{41}\right) = \quad \frac{104}{123}$

Thus, to simplify the quotient of two fractions, we multiply numerator by the inverse of the denominator," i.e.,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Be careful not to confuse what is divided by what: the expression $\dfrac{\frac{a}{b}}{c}$ is ambiguous:

Does it mean: $\dfrac{\left(\frac{a}{b}\right)}{c}$ or $\dfrac{a}{\left(\frac{b}{c}\right)}$?

These two expressions mean very different things:

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \left(\frac{a}{b}\right)\left(\frac{1}{c}\right) = \frac{a \cdot 1}{bc} = \frac{a}{bc}$$

and

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{\left(\frac{a}{1}\right)}{\left(\frac{b}{c}\right)} = \left(\frac{a}{1}\right)\left(\frac{c}{b}\right) = \frac{ac}{b}$$

Example 13:

$\frac{\left(\frac{1}{2}\right)}{3} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) =$ $\frac{1}{6}$ (Remember that pizza?)

Example 14: $\frac{1}{\left(\frac{2}{3}\right)} =$ $\frac{3}{2}$