

Algebra of Fractions

Multiplication

It is very easy to multiply two fractions together: simply multiply together the numerators and denominators:

$$\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

or, using the dot notation for multiplication:

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Example 1: $\left(\frac{x+1}{x+2}\right) \times \left(\frac{x+7}{x-3}\right) = \frac{(x+1) \times (x+7)}{(x+2) \times (x-3)} =$

$$\frac{x^2 + 8x + 7}{x^2 - x - 6}$$

or, without \times signs:

$$\left(\frac{x+1}{x+2}\right) \left(\frac{x+7}{x-3}\right) = \frac{(x+1)(x+7)}{(x+2)(x-3)} = \frac{x^2 + 8x + 7}{x^2 - x - 6}$$

Example 2: Combine $\left(\frac{2+x}{x}\right) \cdot \left(\frac{2-x}{3+x}\right)$ into one fraction.

Solution: $\left(\frac{2+x}{x}\right) \cdot \left(\frac{2-x}{3+x}\right) = \frac{(2+x)(2-x)}{x(3+x)} = \frac{4-x^2}{3x+x^2}$

Example 3: Combine $\left(\frac{9+x^2}{x^2+3}\right) \cdot \left(\frac{9-x^2}{3+x}\right)$ into one fraction.

Solution: $\left(\frac{9+x^2}{x^2+3}\right) \cdot \left(\frac{9-x^2}{3+x}\right)$ = (factoring $9-x^2$)

$$\frac{(9+x^2)(3-x)(3+x)}{(x^2+3)(3+x)} = \frac{(9+x^2)(3-x)}{(x^2+3)} =$$

$$\frac{9(3-x) + x^2(3-x)}{x^2+3} =$$

$$\frac{9(3) - 9(x) + x^2(3) - x^2(x)}{x^2+3} = \frac{27 - 9x + 3x^2 - x^3}{x^2+3}$$

Suppose $c = x$ and $d = 1$, so that $\frac{c}{d} = \frac{x}{1} = x$. Then $\left(\frac{a}{b}\right) \cdot x = \left(\frac{a}{b}\right) \cdot \left(\frac{x}{1}\right) = \frac{ax}{b(1)} = \frac{ax}{b}$,

so multiplying a fraction by x has the same effect as multiplying the numerator of the fraction by x .

Example 4: $\frac{2x+3}{3x+1}\pi = \frac{(2x+3)\pi}{3x+1}$

Addition

Case 1: Same Denominator:

To add two fractions with the same denominator, *add the numerators together*, i.e.,

$$\frac{a}{h} + \frac{b}{h} = \frac{a+b}{h}$$

(Consider: two-sixths of a pizza plus one-sixth of a pizza equals three-sixths, or one-half of a pizza.)

Case 2: Different Denominators:

For fractions with different denominators, say $\frac{a}{b}$ and $\frac{c}{d}$, we must first alter the fractions to have a **common denominator**, say $h = bd$. We do this by multiplications by 1, where we write 1 as an appropriate fraction:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot 1 + 1 \cdot \frac{c}{d} = \left(\frac{a}{b}\right) \cdot \left(\frac{d}{d}\right) + \left(\frac{b}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

We have thus derived the basic addition formula:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

The formula is easily remembered as “cross-multiplication”:

$$\frac{a}{b} \bowtie \frac{c}{d} \rightarrow \frac{ad + bc}{bd}$$

Example 5:

$$\begin{aligned} \frac{3x + 2}{5} + \frac{7x + 1}{8} &= \\ \frac{(3x + 2)8 + 5(7x + 1)}{5(8)} &= \\ \frac{24x + 16 + 35x + 5}{40} &= \frac{59x + 21}{40} \end{aligned}$$

Example 6:

$$\frac{2x^2}{3} + \frac{6}{5x} = \left(\frac{2x^2}{3}\right)\left(\frac{5x}{5x}\right) + \left(\frac{6}{5x}\right)\left(\frac{3}{3}\right) =$$

$$\frac{10x^3}{15x} + \frac{18}{15x} = \frac{10x^3 + 18}{15x}$$

Factor if You Can

Calculations with fractions can be simplified by factoring if there are common factors:

Example 7:

$$\frac{x^2 - 1}{15x + 3} + \frac{x^2 - x - 2}{5x + 1} = \frac{(x + 1)(x - 1)}{3(5x + 1)} + \frac{(x + 1)(x - 2)}{5x + 1} =$$

$$\frac{x + 1}{5x + 1} \left(\frac{x - 1}{3} + \frac{3(x - 2)}{3} \right) = \frac{x + 1}{5x + 1} \left(\frac{x - 1 + 3x - 6}{3} \right) =$$

$$\frac{x + 1}{5x + 1} \left(\frac{4x - 7}{3} \right) = \frac{(x + 1)(4x - 7)}{3(5x + 1)}$$

Danger: Notice again that $\frac{a}{b} + \frac{c}{d}$ does **NOT** equal $\frac{a + c}{b + d}$!!

Example 8: Combine $\frac{h + 5}{5 - h} + \frac{h}{1 + h}$ into one fraction.

Solution

$$\frac{h + 5}{5 - h} + \frac{h}{1 + h} = \frac{(h + 5)(1 + h) + (5 - h)h}{(5 - h)(1 + h)} =$$

$$\frac{h + h^2 + 5h + 5 + 5h - h^2}{5 + 5h - h - h^2} = \frac{11h + 5}{5 + 4h - h^2}$$

Example 9: Combine $\frac{x + 5h}{x - h} + 3$ into one fraction.

Solution:

$$\frac{x + 5h}{x - h} + 3 = \frac{x + 5h}{x - h} + \frac{3}{1} = \frac{(x + 5h) + 3(x - h)}{x - h} =$$

$$\frac{x + 5h + 3x - 3h}{x - h} = \frac{4x + 2h}{x - h}$$

Note: In the basic equation

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

we used the denominator $h = bd$ because it was a **common denominator** for both $\frac{a}{b}$ and $\frac{c}{d}$; however, in many situations, a smaller common denominator will exist, and computations will be greatly simplified if we use it instead. Here is a good example:

Example 10: Combine $\frac{3 - 2x^2}{3a^4(x + 2)^3(x - 1)^3} + \frac{x + 1}{6a^3(x + 2)^3(x - 1)^2}$

Solution:

We could use the common denominator

$$h = [3a^4(x + 2)^3(x - 1)^3] [6a^3(x + 2)^3(x - 1)^2] = 12a^7(x + 2)^6(x - 1)^5$$

...but that would be inefficient: we notice that there are many common factors. The easier way is to factor out the common factors and then add the simplified fractions:

$$\frac{3 - 2x^2}{3a^4(x + 2)^3(x - 1)^3} + \frac{x + 1}{6a^3(x + 2)^3(x - 1)^2} =$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2} \left[\frac{3-2x^2}{a(x-1)} + \frac{x+1}{2} \right] =$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2} \left[\frac{(3-2x^2)2 + a(x-1)(x+1)}{a(x-1)(2)} \right] =$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2} \left[\frac{6-4x^2+ax^2-a}{2a(x-1)} \right] =$$

$$\frac{(a-4)x^2+6-a}{6a^4(x+2)^3(x-1)^3}$$

It is good practice to perform side calculations: after the first line, it is better to add the simplified fractions in a separate calculation: we would write

$$\frac{3-2x^2}{a(x-1)} + \frac{x+1}{2} = \frac{(3-2x^2)2 + a(x-1)(x+1)}{a(x-1)(2)} = \frac{6-4x^2+ax^2-a}{2a(x-1)} =$$

and then we would write:

$$\frac{3-2x^2}{3a^4(x+2)^3(x-1)^3} + \frac{x+1}{6a^3(x+2)^3(x-1)^2} =$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2} \left[\frac{6-4x^2+ax^2-a}{2a(x-1)} \right] =$$

$$\frac{(a-4)x^2+6-a}{6a^4(x+2)^3(x-1)^3}$$

Subtraction

If you know how to add fractions then you know how to subtract them; the trick is to convert $-\left(\frac{c}{d}\right)$ into $\frac{-c}{d}$:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad + b(-c)}{bd} = \frac{ad - bc}{bd},$$

so we have the subtraction formula

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Example 11:

$$\frac{5x + 1}{7x - 1} - \frac{3x - 2}{4x + 3} = \frac{(5x + 1)(4x + 3) - (7x - 1)(3x - 2)}{(7x - 1)(4x + 3)} =$$
$$\frac{20x^2 + 19x + 3 - (21x^2 - 17x + 2)}{(7x - 1)(4x + 3)} = \frac{-x^2 + 36x + 1}{(7x - 1)(4x + 3)}$$

Example 12:

Combine $\frac{a + b}{a - b} - \frac{a - b}{a + b}$ into one fraction.

Solution:

$$\frac{a + b}{a - b} - \frac{a - b}{a + b} = \frac{(a + b)(a + b) - (a - b)(a - b)}{(a - b)(a + b)} =$$
$$\frac{(a^2 + 2ab + b^2) - (a^2 - 2ab + b^2)}{a^2 - b^2} = \frac{4ab}{a^2 - b^2}$$

Division

To divide $\frac{a}{b}$ by $\frac{c}{d}$, we remember that $\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{cd} = 1$,

and multiply by 1 in a special form:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \cdot \frac{\left(\frac{d}{c}\right)}{\left(\frac{d}{c}\right)} = \frac{\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)}{\left(\frac{c}{d}\right)\left(\frac{d}{c}\right)} = \frac{\left(\frac{ad}{bc}\right)}{1} = \frac{ad}{bc}$$

Our basic division formula is thus

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$$

Thus, to simplify the quotient of two fractions, we multiply numerator by the inverse of the denominator," i.e.,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 13: Simplify $\frac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a^2+b^2}{2}}$

Solution:

$$\begin{aligned} \frac{\frac{a+b}{a-b} + \frac{a-b}{a+b}}{\frac{a^2+b^2}{2}} &= \frac{\frac{(a+b)^2+(a-b)^2}{(a-b)(a+b)}}{\frac{a^2+b^2}{2}} = \frac{\frac{a^2+2ab+b^2+a^2-2ab+b^2}{(a^2-b^2)}}{\frac{a^2+b^2}{2}} = \\ \frac{\frac{2a^2+2b^2}{(a^2-b^2)}}{\frac{a^2+b^2}{2}} &= \left(\frac{2(a^2+b^2)}{(a^2-b^2)}\right) \left(\frac{2}{a^2+b^2}\right) = \frac{4}{a^2-b^2} \end{aligned}$$