Polynomials

A **polynomial** in x is simply an addition of non-negative integer powers of x multiplied by constants. Examples are:

$$x^2 + 5x - 3 \qquad \sqrt{2}x \qquad x^3$$

$$\pi x^8 + 2 \qquad 3 \qquad \frac{1}{2}x - \frac{3}{2}$$

Examples of *non*-polynomials are:

$x^2 + \sqrt{x}$ (because $\sqrt{x} = x^{\frac{1}{2}}$ is a *non-integer* power of x)

$3 + \dfrac{1}{x}$ (because $\frac{1}{x} = x^{-1}$ is a *negative* power of x)

The general expression for a polynomial is

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, called the **coefficients** of $p(x)$.

The **degree** of a polynomial is the highest power of x that it contains. For example, $x^4 - 1$ is of degree 4, while $\sqrt{3}$ is of degree 0. Polynomials of low degrees have special names:

degree	**form**	**name**
0	a	constant
1	$ax + b$	linear($a \neq 0$)
2	$ax^2 + bx + c$	quadratic($a \neq 0$)
3	$ax^3 + bx^2 + cx + d$	cubic($a \neq 0$)

Addition or multiplication of two polynomials will always yield another polynomial. Multiplication can be easily done using "long multiplication". The reverse process of finding how to write a given polynomial as a product of lower degree polynomials is called "factoring" and is very important in Calculus.

When manipulating polynomials it is always best to write them with the highest degree term on the left and the lowest degree term on the right, and with the terms arranged in descending order of degree from left to right.

Multiplication of Algebraic Expressions

Most calculus students are comfortable with the multiplication of simple expressions such as

$$(a+b)(c+d) = ac + ad + bc + bd$$

For multiplication of longer expressions a person can always resort, if confused, to "long multiplication," as we now illustrate.

Example 1 Multiply $3x + 4y - 1$ by $x - 2y + 2$.

Solution: An inefficient way of carrying out this calculation would be:

$(3x + 4y - 1) \times (x - 2y + 2) =$

$(3x + 4y - 1) \times x + (3x + 4y - 1) \times (-2y) + (3x + 4y - 1) \times (2) =$

$(3x^2 + 4xy - x) + (-6xy - 8y^2 + 2y) + (6x + 8y - 2) =$

$3x^2 - 2xy - 8y^2 + 5x + 10y - 2$

A "long multiplication" would be written out as follows:

$$\begin{array}{rrrrrr}
 & & & 3x & +4y & -1 \\
 & & & x & -2y & +2 \\
\hline
 & & & 6x & +8y & -2 \\
 & -6xy & -8y^2 & & +2y & \\
3x^2 & +4xy & & -x & & \\
\hline
3x^2 & -2xy & -8y^2 & +5x & +10y & -2
\end{array}$$

The procedure is just that of ordinary multiplication of numbers; the only difference is in the placement of "like terms" underneath each other to aid in the final addition.

Certain combinations of terms appear so often that you are well-advised to memorize them:

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Multiplication of Polynomials

Example 2: Multiply $3x^3 + 4x^2 - 1$ by $x^2 - 2x + 2$.

Solution: A "long multiplication" would be written out as follows:

$$\begin{array}{rrrrrr}
 & & 3x^3 & +4x^2 & & -1 \\
 & & & x^2 & -2x & +2 \\
\hline
 & & 6x^3 & +8x^2 & & -2 \\
 & -6x^4 & -8x^3 & & +2x & \\
3x^5 & +4x^4 & & -x^2 & & \\
\hline
3x^5 & -2x^4 & -2x^3 & +7x^2 & +2x & -2
\end{array}$$

Detached Coefficients

Multiplication of polynomials can be simplified by “detaching the coefficients” from the powers of x: we repeat the above calculation:

x^5	x^4	x^3	x^2	x^1	x^0
		3	4	0	−1
			1	−2	2
		6	8	0	−2
	−6	−8	0	2	0
3	4	0	−1	0	0
3	−2	−2	7	2	−2

which we interpret to mean
$(3x^3 + 4x^2 - 1) \times (x^2 - 2x + 2) =$

$3x^5 - 2x^4 - 2x^3 + 7x^2 + 2x - 2$

Powers of $a + b$ — Pascal’s Triangle

This is an easy way of finding integer powers of $a + b$. The triangle is bordered by 1’s and has the property that every interior entry is the sum of the two entries just above it:

$n = 0$	1							
$n = 1$	1	1						
$n = 2$	1	2	1					
$n = 3$	1	3	3	1				
$n = 4$	1	4	6	4	1			
$n = 5$	1	5	10	10	5	1		
$n = 6$	1	6	15	20	15	6	1	
$n = 7$	1	7	21	35	35	21	7	1

The n^{th} row of the triangle contains the coefficients of $(a + b)^n$:

$n = 0$: $(a + b)^0 = 1 \cdot a^0 b^0 = 1$

$n = 1$: $(a + b)^1 = 1 \cdot a^1 b^0 + 1 \cdot a^0 b^1 = a + b$

$n = 2$: $(a + b)^2 = 1 \cdot a^2 b^0 + 2 \cdot a^1 b^1 + 1 \cdot a^0 b^2 = a^2 + 2ab + b^2$

$n = 3$: $(a + b)^3 = 1 \cdot a^3 b^0 + 3 \cdot a^2 b^1 + 3 \cdot a^1 b^2 + 1 \cdot a^0 b^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

$n = 4$: $(a+b)^4 = 1 \cdot a^4 b^0 + 4 \cdot a^3 b^1 + 6 \cdot a^2 b^2 + 4 \cdot a^1 b^3 + 1 \cdot a^0 b^4 =$
$a^4 + 4a^3 b + 6a^2 b^2 + 4a^1 b^3 + b^4$

$n = 5$: $(a+b)^5 = 1 \cdot a^5 b^0 + 5 \cdot a^4 b^1 + 10 \cdot a^3 b^2 + 10 \cdot a^2 b^3 + 5 \cdot a^1 b^4 + 1 \cdot a^0 b^5 =$
$a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + 5ab^4 + b^5$

and so on.

We can build another useful triangle:

$$
\begin{array}{lrrrrrrrr}
(a+b)^0 = & & & & & & & & 1 \\
(a+b)^1 = & & & & & & & a & +b \\
(a+b)^2 = & & & & & & a^2 & +2ab & +b^2 \\
(a+b)^3 = & & & & & a^3 & +3a^2b & +3ab^2 & +b^3 \\
(a+b)^4 = & & & & a^4 & +4a^3b & +6a^2b^2 & +4ab^3 & +b^4 \\
(a+b)^5 = & & & a^5 & +5a^4b & +10a^3b^2 & +10a^2b^3 & +5ab^4 & +b^5 \\
(a+b)^6 = & & a^6 & +6a^5b & +15a^4b^2 & +20a^3b^3 & +15a^2b^4 & +6ab^5 & +b^6 \\
(a+b)^7 = & a^7 & +7a^6b & +21a^5b^2 & +35a^4b^3 & +35a^3b^4 & +21a^2b^5 & +7ab^6 & +b^7
\end{array}
$$

Division

The division of one algebraic expression by another frequently gives people difficulty. However, the technique of "long division" is quite important and is similar to the usual "long division" of decimal numbers.

Example 3:

Divide $x^3 + x^2 + x - 3$ by $x - 1$, i.e., simplify $\dfrac{x^3 + x^2 + x - 3}{x - 1}$

Solution: We'll go through this division very slowly to see precisely what is happening. We start by arranging the divisor $x - 1$ and the dividend $(x^3 + x^2 + x - 3)$ in the usual way:

$$\text{divisor}\,|\overline{\text{dividend}} \qquad\qquad x - 1\,|\overline{x^3 + x^2 + x - 3}$$

Notice that both terms are written with the powers of x in *descending order*.

1. Take the first term x in the divisor and divide it into the first term x^3 in the dividend; the result is x^2, which we write above the x^2 as shown:

$$\begin{array}{r|llll} & & x^2 & & \\ \hline x - 1 & x^3 & +x^2 & +x & -3 \end{array} \qquad \left(= \frac{x^3}{x}\right)$$

Now multiply the divisor $(x - 1)$ by x^2, place the result $(x^3 - x^2)$ under the dividend (with correct positioning of the powers of x), and subtract:

$$
\begin{array}{r|llll l}
 & x^2 & & & & \\
\hline
x-1 & x^3 & +x^2 & +x & -3 & \\
 & x^3 & -x^2 & & & (= x^2(x-1)) \\
\cline{2-3}
 & & 2x^2+x-3 & & & (= x^3+x^2+x-3-x^2(x-1))
\end{array}
$$

Note that the bottom term is the dividend minus x^2 times the divisor:

$2x^2 + x - 3 = (x^3 + x^2 + x - 3) - x^2(x - 1)$

It will be referred to as the “new” or “second” dividend”. We operate on it just as we did on the original dividend: divide the x in the divisor into the $2x^2$ of the new dividend, and place the resulting $2x$ above the division line as shown:

$$
\begin{array}{r|llll}
 & x^2 & +2x & & \\
\hline
x-1 & x^3 & +x^2 & +x & -3 \\
 & x^3 & -x^2 & & \\
\cline{2-3}
 & & 2x^2 & +x & -3
\end{array}
\qquad \left(= \frac{2x^2}{x}\right)
$$

Next we multiply the divisor $(x - 1)$ by $2x$ and place the resulting $2x^2 - 2x$ under the new dividend:

$$
\begin{array}{r|llll l}
 & x^2 & +2x & & & \\
\hline
x-1 & x^3 & +x^2 & +x & -3 & \\
 & x^3 & -x^2 & & & \\
\cline{2-3}
 & & 2x^2 & +x & -3 & \\
 & & 2x^2 & -2x & & = 2x(x-1) \\
\cline{3-4}
\end{array}
$$

Next we subtract:

$$
\begin{array}{r|rrrrl}
 & & x^2 & +2x & & \\ \hline
x-1 & x^3 & +x^2 & +x & -3 & \\
 & x^3 & -x^2 & & & \\ \cline{2-3}
 & & 2x^2 & +x & -3 & \\
 & & 2x^2 & -2x & & \\ \cline{3-4}
 & & & 3x-3 & & = 2x^2+x-3-(2x^2-2x)
\end{array}
$$

Note that the bottom term is now the dividend minus $x^2 + 2x$ times the divisor:

$3x - 3 = (x^3 + x^2 + x - 3) - (x^2 + 2x)(x - 1)$

3. This bottom term is the “second” or “even newer dividend”! and by now you should be able to guess what to do with it. But, if you’re still not sure: divide x in the divisor into the $3x$ of the “even newer dividend,” and place the resulting 3 above the division line. Then multiply the divisor $x - 1$ by 3, and subtract the result from the “even newer dividend”

$$
\begin{array}{r|rrrrl}
 & & x^2 & +2x & +3 & \left(=\dfrac{3x}{x}\right) \\ \hline
x-1 & x^3 & +x^2 & +x & -3 & \\
 & x^3 & -x^2 & & & \\ \cline{2-3}
 & & 2x^2 & +x & -3 & \\
 & & 2x^2 & -2x & & \\ \cline{3-4}
 & & & 3x & -3 & \\
 & & & 3x & -3 & (= 3(x-1)) \\ \cline{4-5}
 & & & & 0 &
\end{array}
$$

Now the bottom term 0 is the dividend minus $x^2 + 2x + 3$ times the divisor:

$0 = (x^3 + x^2 + x - 3) - (x^2 + 2x + 3)(x - 1)$

so we have the equation

$(x^3 + x^2 + x - 3) = (x^2 + 2x + 3)(x - 1)$, a factorization of the dividend.

Thus we have $\dfrac{x^3 + x^2 + x - 3}{x - 1} = x^2 + 2x + 3.$

Detached Coefficients

As with multiplication, division of polynomials can be simplified by "detaching the coefficients" from the powers of x: we repeat the above calculation:

$$
\begin{array}{rr|rrrr}
x^1 & x^0 & x^3 & x^2 & x^1 & x^0 \\
\hline
 & & & 1 & 2 & 3 \\
\cline{3-6}
1 & -1 & 1 & 1 & 1 & -3 \\
 & & 1 & -1 & & \\
\cline{3-4}
 & & & 2 & 1 & -3 \\
 & & & 2 & -2 & \\
\cline{4-5}
 & & & & 3 & -3 \\
 & & & & 3 & -3 \\
\cline{5-6}
 & & & & & 0
\end{array}
$$

which means exactly the same thing as the previous calculation!

Example 4

Divide $x^3 + 2x^2 + 3x + 2$ by $x^2 + 1$, i.e., simplify $\dfrac{x^3 + 2x^2 + 3x + 2}{x^2 + 1}$.

Solution: Again we set up the divisor and dividend with the terms in decreasing powers of x.

$$
x^2 + 1 \;\overline{\Big|\; x^3 \quad +2x^2 \quad +3x \quad +2}
$$

1. Divide x^2 into x^3 to get x; multiply $x^2 + 1$ by x and subtract the result from the dividend:

$$\begin{array}{r|rrrr} & & & x & \\ \hline x^2+1 & x^3 & +2x^2 & +3x & +2 \\ & x^3 & & +x & \\ \cline{2-4} & & 2x^2 & +2x & +2 \end{array}$$

Note that the bottom term is the dividend minus x times the divisor:

$2x^2 + 2x + 2 = (x^3 + 2x^2 + 3x + 2) - x(x^2 + 1)$

2. Divide x^2 into $2x^2$ to get 2; multiply $x^2 + 1$ by 2 and subtract the result from the "new dividend".

$$\begin{array}{r|rrrr} & & & x & +2 \\ \hline x^2+1 & x^3 & +2x^2 & +3x & +2 \\ & x^3 & & +x & \\ \cline{2-4} & & 2x^2 & +2x & +2 \\ & & 2x^2 & & +2 \\ \cline{3-5} & & & 2x & \end{array}$$

Now the bottom term is the dividend minus $x + 2$ times the divisor:$2x = (x^3 + 2x^2 + 3x + 2) - (x + 2)(x^2 + 1)$

There is a major difference from the previous example: The remainder term is non zero. This means that the divisor is not a factor of the dividend. We can use the last equation to write our simplified expression:

$$\frac{x^3+2x^2+3x+2}{x^2+1} = x+2+\frac{2x}{x^2+1}$$

The detached coefficient calculation is:

$$\begin{array}{ccc|cccc}
x^2 & x^1 & x^0 & x^3 & x^2 & x^1 & x^0 \\
\hline
 & & & & & 1 & 2 \\
\cline{4-7}
1 & 0 & 1 & 1 & 2 & 3 & 2 \\
 & & & 1 & 0 & 1 & \\
\cline{4-6}
 & & & & 2 & 2 & 2 \\
 & & & & 2 & 0 & 2 \\
\cline{5-7}
 & & & & & 2 & 0
\end{array}$$

Example 5 Divide $2t^2 - t + t^3$ by $t+1$

Solution: We have to rewrite the dividend as $t^3 + 2t^2 - t$ so that the powers of t are in decreasing order. The solution is:

$$\begin{array}{r|rrrr}
 & & t^2 & +t & -2 \\
\cline{2-5}
t+1 & t^3 & +2t^2 & -t & \\
 & t^3 & +t^2 & & \\
\cline{2-4}
 & & t^2 & -t & \\
 & & t^2 & +t & \\
\cline{3-5}
 & & & -2t & \\
 & & & -2t & -2 \\
\cline{4-5}
 & & & & 2
\end{array}$$

so we have

$$\frac{t^3+2t^2-t}{t+1} = t^2+t-2+\frac{2}{t+1}$$

The detached coefficient calculation is:

t^1	t^0	$\|t^3$	t^2	t^1	t^0
			1	1	-2
1	1	1	2	-1	
		1	1		
			1	-1	
			1	1	
				-2	
				-2	-2
					2