Algebra of Fractions

Multiplication

It is very easy to multiply two fractions together: simply multiply together the numerators and denominators:

$$\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{a \times c}{b \times d} = \frac{ac}{bd}$$

or, using the dot notation for multiplication:

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Example 1: $\left(\frac{x+1}{x+2}\right) \times \left(\frac{x+7}{x-3}\right) = \frac{(x+1) \times (x+7)}{(x+2) \times (x-3)} =$

$$\frac{x^2 + 8x + 7}{x^2 - x - 6}$$

or, without $\times$ signs:

$$\left(\frac{x+1}{x+2}\right)\left(\frac{x+7}{x-3}\right) = \frac{(x+1)(x+7)}{(x+2)(x-3)} = \frac{x^2 + 8x + 7}{x^2 - x - 6}$$

Example 2: Combine $\left(\frac{2+x}{x}\right) \cdot \left(\frac{2-x}{3+x}\right)$ into one fraction.

Solution: $\left(\frac{2+x}{x}\right) \cdot \left(\frac{2-x}{3+x}\right) = \frac{(2+x)(2-x)}{x(3+x)} = \frac{4 - x^2}{3x + x^2}$

Example 3: Combine $\left(\frac{9+x^2}{x^2+3}\right)\cdot\left(\frac{9-x^2}{3+x}\right)$ into one fraction.

Solution: $\left(\frac{9+x^2}{x^2+3}\right)\cdot\left(\frac{9-x^2}{3+x}\right)$ =(factoring $9-x^2$)

$$\frac{(9+x^2)(3-x)(3+x)}{(x^2+3)(3+x)} = \frac{(9+x^2)(3-x)}{(x^2+3)} =$$

$$\frac{9(3-x)+x^2(3-x)}{x^2+3} =$$

$$\frac{9(3)-9(x)+x^2(3)-x^2(x)}{x^2+3} = \boxed{\frac{27-9x+3x^2-x^3}{x^2+3}}$$

Suppose $c = x$ and $d = 1$, so that $\frac{c}{d} = \frac{x}{1} = x$. Then $\left(\frac{a}{b}\right)\cdot x = \left(\frac{a}{b}\right)\cdot\left(\frac{x}{1}\right) = \frac{ax}{b(1)} = \frac{ax}{b}$,

so multiplying a fraction by x has the same effect as multiplying the numerator of the fraction by x.

Example 4: $\frac{2x+3}{3x+1}\pi = \boxed{\frac{(2x+3)\pi}{3x+1}}$

Addition

Case 1: Same Denominator:

To add two fractions with the same denominator, *add the numerators together*, i.e.,

$$\boxed{\frac{a}{h}+\frac{b}{h}=\frac{a+b}{h}}$$

(Consider: two-sixths of a pizza plus one-sixth of a pizza equals three-sixths, or one-half of a pizza.)

Case 2: Different Denominators:

For fractions with different denominators, say $\frac{a}{b}$ and $\frac{c}{d}$, we must first alter the fractions to have a **common denominator**, say $h = bd$.

We do this by multiplications by 1, where we write 1 as an appropriate fraction:

$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot 1+1\cdot\frac{c}{d}=\left(\frac{a}{b}\right)\cdot\left(\frac{d}{d}\right)+\left(\frac{b}{b}\right)\cdot\left(\frac{c}{d}\right)=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

We have thus derived the basic addition formula:

$$\boxed{\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}}$$

The formula is easily remembered as "cross-multiplication":

$$\frac{a}{b}\bowtie\frac{c}{d}\rightarrow\frac{ad+bc}{bd}$$

Example 5:

$$\frac{3x+2}{5}+\frac{7x+1}{8}=$$

$$\frac{(3x+2)8+5(7x+1)}{5(8)}=$$

$$\frac{24x+16+35x+5}{40}=\boxed{\frac{59x+21}{40}}$$

Example 6:

$$\frac{2x^2}{3} + \frac{6}{5x} = \left(\frac{2x^2}{3}\right)\left(\frac{5x}{5x}\right) + \left(\frac{6}{5x}\right)\left(\frac{3}{3}\right) =$$

$$\frac{10x^3}{15x} + \frac{18}{15x} = \boxed{\frac{10x^3 + 18}{15x}}$$

Factor if You Can

Calculations with fractions can be simplified by factoring if there are common factors:

Example 7:

$$\frac{x^2 - 1}{15x + 3} + \frac{x^2 - x - 2}{5x + 1} = \frac{(x+1)(x-1)}{3(5x+1)} + \frac{(x+1)(x-2)}{5x+1} =$$

$$\frac{x+1}{5x+1}\left(\frac{x-1}{3} + \frac{3(x-2)}{3}\right) = \frac{x+1}{5x+1}\left(\frac{x - 1 + 3x - 6}{3}\right) =$$

$$\frac{x+1}{5x+1}\left(\frac{4x-7}{3}\right) = \boxed{\frac{(x+1)(4x-7)}{3(5x+1)}}$$

Danger:

Notice again that $\frac{a}{b} + \frac{c}{d}$ does **NOT** equal $\frac{a+c}{b+d}$!!

Example 8:

Combine $\frac{h+5}{5-h} + \frac{h}{1+h}$ into one fraction.

Solution

$$\frac{h+5}{5-h} + \frac{h}{1+h} = \frac{(h+5)(1+h) + (5-h)h}{(5-h)(1+h)} =$$

$$\frac{h + h^2 + 5h + 5 + 5h - h^2}{5 + 5h - h - h^2} = \boxed{\frac{11h + 5}{5 + 4h - h^2}}$$

Example 9: Combine $\frac{x+5h}{x-h}+3$ into one fraction.

Solution:

$$\frac{x+5h}{x-h}+3=\frac{x+5h}{x-h}+\frac{3}{1}=\frac{(x+5h)+3(x-h)}{x-h}=$$

$$\frac{x+5h+3x-3h}{x-h}=\boxed{\frac{4x+2h}{x-h}}$$

Note: In the basic equation

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

we used the denominator $h=bd$ because it was a **common denominator** for both $\frac{a}{b}$ and $\frac{c}{d}$; however, in many situations, a smaller common denominator will exist, and computations will be greatly simplified if we use it instead. Here is a good example:

Example 10: Combine $\frac{3-2x^2}{3a^4(x+2)^3(x-1)^3}+\frac{x+1}{6a^3(x+2)^3(x-1)^2}$

Solution:
We could use the common denominator
$h=\left[3a^4(x+2)^3(x-1)^3\right]\left[6a^3(x+2)^3(x-1)^2\right]=12a^7(x+2)^6(x-1)^5$
...but that would be inefficient: we notice that there are many common factors. The easier way is to factor out the common factors and then add the simplified fractions:

$$\frac{3-2x^2}{3a^4(x+2)^3(x-1)^3}+\frac{x+1}{6a^3(x+2)^3(x-1)^2}=$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2}\left[\frac{3-2x^2}{a(x-1)}+\frac{x+1}{2}\right]=$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2}\left[\frac{(3-2x^2)2+a(x-1)(x+1)}{a(x-1)(2)}\right]=$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2}\left[\frac{6-4x^2+ax^2-a}{2a(x-1)}\right] =$$

$$\frac{(a-4)x^2+6-a}{6a^4(x+2)^3(x-1)^3}$$

It is good practice to perform side calculations: after the first line, it is better to add the simplified fractions in a separate calculation: we would write

$$\frac{3-2x^2}{a(x-1)} + \frac{x+1}{2} = \frac{(3-2x^2)2 + a(x-1)(x+1)}{a(x-1)(2)} = \frac{6-4x^2+ax^2-a}{2a(x-1)} =$$

and then we would write:

$$\frac{3-2x^2}{3a^4(x+2)^3(x-1)^3} + \frac{x+1}{6a^3(x+2)^3(x-1)^2} =$$

$$\frac{1}{3a^3(x+2)^3(x-1)^2}\left[\frac{6-4x^2+ax^2-a}{2a(x-1)}\right] =$$

$$\frac{(a-4)x^2+6-a}{6a^4(x+2)^3(x-1)^3}$$

Subtraction

If you know how to add fractions then you know how to subtract them; the trick is to convert $-\left(\frac{c}{d}\right)$ into $\frac{-c}{d}$:

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad + b(-c)}{bd} = \frac{ad-bc}{bd},$$

so we have the subtraction formula $\boxed{\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad-bc}{bd}}$

Example 11:

$$\frac{5x+1}{7x-1} - \frac{3x-2}{4x+3} = \frac{(5x+1)(4x+3)-(7x-1)(3x-2)}{(7x-1)(4x+3)} =$$

$$\frac{20x^2+19x+3-(21x^2-17x+2)}{(7x-1)(4x+3)} = \boxed{\frac{-x^2+36x+1}{(7x-1)(4x+3)}}$$

Example 12:

Combine $\frac{a+b}{a-b} - \frac{a-b}{a+b}$ into one fraction.

Solution:

$$\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)(a+b)-(a-b)(a-b)}{(a-b)(a+b)} =$$

$$\frac{(a^2+2ab+b^2)-(a^2-2ab+b^2)}{a^2-b^2} = \boxed{\frac{4ab}{a^2-b^2}}$$

Division

To divide $\frac{a}{b}$ by $\frac{c}{d}$, we remember that $\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{cd} = 1$,

and multiply by 1 in a special form:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \cdot \frac{\left(\frac{d}{c}\right)}{\left(\frac{d}{c}\right)} = \frac{\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)}{\left(\frac{c}{d}\right)\left(\frac{d}{c}\right)} = \frac{\left(\frac{ad}{bc}\right)}{1} = \frac{ad}{bc}$$

Our basic division formula is thus

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{ad}{bc}$$

Thus, to simplify the quotient of two fractions, we multiply numerator by the inverse of the denominator," i.e.,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Example 13: Simplify $\frac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a^2+b^2}{2}}$

Solution:

$$\frac{\frac{a+b}{a-b}+\frac{a-b}{a+b}}{\frac{a^2+b^2}{2}} = \frac{\frac{(a+b)^2+(a-b)^2}{(a-b)(a+b)}}{\frac{a^2+b^2}{2}} = \frac{\frac{a^2+2ab+b^2+a^2-2ab+b^2}{(a^2-b^2)}}{\frac{a^2+b^2}{2}} =$$

$$\frac{\frac{2a^2+2b^2}{(a^2-b^2)}}{\frac{a^2+b^2}{2}} = \left(\frac{2(a^2+b^2)}{(a^2-b^2)}\right)\left(\frac{2}{a^2+b^2}\right) = \boxed{\frac{4}{a^2-b^2}}$$