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$$2x^2 + x - 1 = (2x - 1)(x + 1)$$

$$8x^2 - 2x - 15 = (2x - 3)(4x + 5)$$

$$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$$

Factoring-2

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$$2x^3 - \frac{17}{3}x^2 - \frac{5}{3}x + 2 = (x-3)(2x-1)\left(x + \frac{2}{3}\right)$$

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$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^4 - 16 = (x-2)(x+2)(x^2+4)$$

Factoring a polynomial is not always a pleasant or easy operation, but it is important: because in addition to telling us where a polynomial is zero, it is the tool we will need to determine exactly where a polynomial is positive or negative. This in turn will vital to the sketching of the graphs of polynomials, one of the highlights of this course.

We will study the procedure in some depth, first for quadratic (or degree 2) polynomials, and then for general polynomials.

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Factoring-5

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so we can either take $a = -2$ and $b = 3$ or $a = 3$ and $b = -2$.

Either way, we get the factorizations

$$x^2 + x - 6 = (x + 2)(x - 3) = (x - 3)(x + 2).$$

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$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

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Thus $2x^2 + x - 6 = 2(x - (-2))\left(x - \frac{3}{2}\right)$

which appears to be different from the factorization $2x^2 + x - 6 = (2x + (-3))(x + 2) = (2x - 3)(x + 2)$ that we found by "inspection"!

However, we note that

$$2(x - (-2))\left(x - \frac{3}{2}\right) = (x + 2)2\left(x - \frac{3}{2}\right) =$$

$$(x + 2)\left(2x - 2\frac{3}{2}\right) = (x + 2)(2x - 3) = (2x - 3)(x + 2),$$

so we have two different but equivalent factorizations of the same polynomial.

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Thus $3x^2 - 5x + 1 = 3\left(x - \frac{5 + \sqrt{13}}{6}\right)\left(x - \frac{5 - \sqrt{13}}{6}\right)$

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Thus $x^2 - 2x + 2 = (x - (1 + i))\,(x - (1 - i))$

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Note that $x^2 + 4$ is irreducible because it has no real roots.

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Example 7: Factor the third degree polynomial $p(x) = x^3 - 5x^2 + 6x - 2$.

Solution: If this polynomial has any integer roots they must divide 2. Since the only integers which divide 2 are -2, -1, 1, and 2, it is useful to construct a table of values:

Factoring-22

x	-2	-1	1	2
$p(x)$	-42	-14	0	-2

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$$
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\hline
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 & x^3 & -x^2 & & \\
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so we have $x^3-5x^2+6x-2=(x-1)(x^2-4x+2)$ and now we find the roots of the quadratic x^2-4x+2 using the quadratic formula with $a=1$, $b=-4$, $c=2$:

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The Rational Root Test

Suppose $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with *integer* coefficients, and $r = \frac{m}{q}$ is a rational number expressed in lowest terms.

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Notice that this only says that $\dfrac{m}{q}$ *can* be a root, it does not say that it *is*!

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Solution: Suppose $\frac{m}{q}$ is a root. Then m divides 2 and q divides 3, so the possibilities are $m = -2, -1, 1, 2$, and $q = -3, -1, 1$, and 3, leaving us with 16 possible values for $\frac{m}{q}$, as shown in the following table.

	$m = -2$	$m = -1$	$m = 1$	$m = 2$
$q = -3$	$\frac{m}{q} = \frac{-2}{-3} = \frac{2}{3}$	$\frac{m}{q} = \frac{-1}{-3} = \frac{1}{3}$	$\frac{m}{q} = \frac{1}{-3} = -\frac{1}{3}$	$\frac{m}{q} = \frac{2}{-3} = -\frac{2}{3}$
$q = -1$	$\frac{m}{q} = \frac{-2}{-1} = 2$	$\frac{m}{q} = \frac{-1}{-1} = 1$	$\frac{m}{q} = \frac{1}{-1} = -1$	$\frac{m}{q} = \frac{2}{-1} = -2$
$q = 1$	$\frac{m}{q} = \frac{-2}{1} = -2$	$\frac{m}{q} = \frac{-1}{1} = -1$	$\frac{m}{q} = \frac{1}{1} = 1$	$\frac{m}{q} = \frac{2}{1} = 2$
$q = 3$	$\frac{m}{q} = \frac{-2}{3} = -\frac{2}{3}$	$\frac{m}{q} = \frac{-1}{3} = -\frac{1}{3}$	$\frac{m}{q} = \frac{1}{3}$	$\frac{m}{q} = \frac{2}{3}$

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x	-2	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	2

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x	-2	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1	2
$p(x)$	-56	-4	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{9}$	0	-2	-4

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\begin{array}{r|rrrr}
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3x-2 & 3x^3 & -8x^2 & +x & +2 \\
 & 3x^3 & -2x^2 & & \\
\hline
 & & -6x^2 & +x & \\
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$(3x - 2)(x^2 - 2x - 1) = 3\left(x - \frac{2}{3}\right)(x^2 - 2x - 1)$.

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$$x^2 - 2x - 1 = \left((x - (1+\sqrt{2})\right)\left((x - (1-\sqrt{2})\right)$$

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$$3\left(x-\frac{2}{3}\right)(x-(1+\sqrt{2}))\,(x-(1-\sqrt{2}))$$