Completing the Square

With quadratic polynomials it is often important to **complete the square.** As a matter of fact, the Quadratic Formula for the roots of $ax^2 + bx + c = 0$ is derived by using this method.

The basic idea is that we rewrite $ax^2 + bx + c$ in the form

$$a[(x + h)^2 + k].$$

Example 1: $x^2 + 2x + 2 = \overbrace{x^2 + 2x + 1}^{(x+1)^2} + 1 = (x+1)^2 + 1$

What we try to do is rearrange the expression so that its first three terms are a perfect square. An organized approach to this is to add an expression that equals 0 in just the right way:

$$x^2 + 2x + 2 = x^2 + 2x + \underbrace{(1 - 1)}_{0} + 2 = \overbrace{x^2 + 2x + 1}^{(x+1)^2} + (-1) + 2 =$$

$(x+1)^2 + 1$

But how did we know to add 0 in the form $1 - 1$? The answer is that 1 is one-half of the coefficient 2 of x in the expression $x^2 + 2x + 2$.

Things get more complicated when the coefficient of x isn't equal to 2:

Example 2:

$$x^2 + 4x + 2 = x^2 + 2(2)x + \underbrace{(2^2 - 2^2)}_{0} + 2 =$$

$$\overbrace{x^2 + 2(2)x + 2^2}^{(x+2)^2} - 2^2 + 2 =$$

$$(x + 2)^2 - 2$$

Here the coefficient of x was 4, so we divided it by 2 to get 2, and then added 0 in the form $2^2 - 2^2$.

When we are dealing with an expression like $x^2 + bx + c$, we have to add 0 in the form of the square of half of b minus itself:

$$x^2 + bx + c = x^2 + 2\left(\frac{b}{2}\right)x + c =$$

$$x^2 + 2\left(\frac{b}{2}\right)x + \underbrace{\left(\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right)}_{0} + c =$$

$$\overbrace{x^2 + 2\left(\frac{b}{2}\right)x + \left(\frac{b}{2}\right)^2}^{\left(x+\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Example 3:

$$x^2 + 5x + 3 = x^2 + 2\left(\frac{5}{2}\right)x + \underbrace{\left(\left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right)}_{0} + 3$$

$$\overbrace{x^2 + 2\left(\frac{5}{2}\right)x + \left(\frac{5}{2}\right)^2}^{\left(x+\frac{5}{2}\right)^2} - \left(\frac{5}{2}\right)^2 + 3\left(\frac{4}{4}\right) = \left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{12}{4} =$$

$$\left(x + \frac{5}{2}\right)^2 - \frac{13}{4}$$

Example 4:

$$x^2 + 9x + 5 = x^2 + 2\left(\frac{9}{2}\right)x + \underbrace{\left(\left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2\right)}_{0} + 5$$

$$\overbrace{x^2 + 2\left(\frac{9}{2}\right)x + \left(\frac{9}{2}\right)^2}^{\left(x+\frac{9}{2}\right)^2} - \left(\frac{9}{2}\right)^2 + 5\left(\frac{4}{4}\right) = \left(x + \frac{9}{2}\right)^2 - \frac{81}{4} + \frac{20}{4} =$$

$$\left(x + \frac{9}{2}\right)^2 - \frac{61}{4}$$

Things get even more complicated when the coefficient a of x^2 is not equal to 1: we have to factor it out before proceeding as above:

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] =$$

$$a\left[x^2 + 2\left(\frac{b}{2a}\right)x + \underbrace{\left(\left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right)}_{0} + \frac{c}{a}\right] =$$

$$a\left[\overbrace{x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2}^{\left(x+\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] =$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2}\right] =$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

Example 5:

$$3x^2 + 5x + 2 = 3\left[x^2 + \frac{5}{3}x + \frac{2}{3}\right] =$$

$$3\left[x^2 + 2\left(\frac{5}{6}\right)x + \underbrace{\left(\left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right)}_{0} + \frac{2}{3}\right] =$$

$$3\left[\overbrace{x^2 + 2\left(\frac{5}{6}\right)x + \left(\frac{5}{6}\right)^2}^{\left(x+\frac{5}{6}\right)^2} - \left(\frac{5}{6}\right)^2 + \frac{2}{3}\left(\frac{12}{12}\right)\right] =$$

$$3\left[\left(x + \frac{5}{6}\right)^2 - \frac{25}{36} + \frac{24}{36}\right] = 3\left[\left(x + \frac{5}{6}\right)^2 - \frac{1}{36}\right] =$$

$$3\left[\left(x + \frac{5}{6}\right)^2 - \left(\frac{1}{6}\right)^2\right] =$$

$$3\left(x + \frac{5}{6} + \frac{1}{6}\right)\left(x + \frac{5}{6} - \frac{1}{6}\right) = 3\,(x + 1)\left(x + \frac{2}{3}\right) =$$

$$(x+1)(3x+2)$$

We return to the general situation. By completing squares, we derived the equation:

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

We use this to solve the equation $ax^2 + bx + c = 0$: since $a \neq 0$ we have

$$\begin{aligned}
ax^2 + bx + c = 0 \quad &\iff \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0 \quad &\iff \\
\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad &\iff \left(\text{ if } b^2 - 4ac \geq 0\right) \\
x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad &\iff \\
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

which is the familiar Quadratic Formula for the roots of the general quadratic equation.